Skolem-type Difference Sets for Cycle Systems

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Definition A **Skolem sequence of order** t is a sequence $S = (s_1, s_2, \ldots, s_{2t})$ of $2t$ integers such that

(S1) for all $k \in \{1, 2, \ldots, t\}$, there exists unique $s_i, s_j \in S$ with $s_i = s_j = k$, and

(S2) if $s_i = s_j = k$ with $i < j$, then $j - i = k$.

Examples

$\boxed{t = 1} \Rightarrow S = (1, 1)$

$\boxed{t = 4} \Rightarrow S = (1, 1, 3, 4, 2, 3, 2, 4)$

$\boxed{t = 5} \Rightarrow S = (2, 4, 2, 3, 5, 4, 3, 1, 1, 5)$

A Skolem sequence of order t

- can be written as a collection of ordered pairs $\{(a_i, b_i) \mid 1 \leq i \leq t, b_i - a_i = i\}$ with $\cup_{i=1}^{t}\{a_i, b_i\} = \{1, 2, \ldots, 2t\}$.
- gives a partition of the set $\{1, 2, \ldots, 3t\}$ into triples (a_i, b_i, c_i) such that $a_i + b_i = c_i$ for each $i = 1, 2, \ldots, t$.

Examples

$\boxed{t = 4} \Rightarrow S = (1, 1, 3, 4, 2, 3, 2, 4)$

$\boxed{t = 5} \Rightarrow S = (2, 4, 2, 3, 5, 4, 3, 1, 1, 5)$

Theorem (Skolem, 1957) A Skolem sequence of order t exists if and only if $t \equiv 0, 1 \pmod 4$.

Definition A **hooked Skolem sequence of order** t is a sequence $HS = (s_1, s_2, \ldots, s_{2t+1})$ of $2t + 1$ integers such that

(S1) for all $k \in \{1, 2, \ldots, t\}$, there exists unique $s_i, s_j \in S$ with $s_i = s_j = k$,

(S2) if $s_i = s_j = k$ with $i < j$, then $j - i = k$, and

(S3) $s_{2t} = 0$.

Examples

$\boxed{t = 2} \Rightarrow HS = (1, 1, 2, 0, 2)$

$\boxed{t = 3} \Rightarrow HS = (3, 1, 1, 3, 2, 0, 2)$

A **hooked Skolem sequence of order** t gives a partition of the set $\{1, 2, \ldots, 3t+1\} \setminus \{3t\}$ into triples (a_i, b_i, c_i) such that $a_i + b_i = c_i$ for each $i = 1, 2, \ldots, t$.

Examples

$\boxed{t = 2} \Rightarrow HS = (1, 1, 2, 0, 2)$

$\boxed{t = 3} \Rightarrow HS = (3, 1, 1, 3, 2, 0, 2)$

Theorem (O'Keefe, 1961) A hooked Skolem sequence of order t exists if and only if $t \equiv 2, 3 \pmod 4$.

__**Definition**__ A **Langford sequence of order** t **and defect** d is a sequence $L = (\ell_1, \ell_2, \ldots, \ell_{2t})$ of $2t$ integers such that

(L1) for all $k \in \{d, d+1, d+2, \ldots, d+t-1\}$, there exists unique $\ell_i, \ell_j \in L$ with $\ell_i = \ell_j = k$, and

(L2) if $\ell_i = \ell_j = k$ with $i < j$, then $j - i = k$.

A **hooked Langford sequence of order** t **and defect** d is a sequence $HL = (\ell_1, \ell_2, \ldots, \ell_{2t+1})$ of $2t + 1$ integers satisfying **(L1)** and **(L2)** above and

(L3) $\ell_{2t} = 0$.

Examples

$t = 5,\ d = 3$

$$L = (7, 5, 3, 6, 4, 3, 5, 7, 4, 6)$$

$t = 5,\ d = 2$

$$HL = (3, 4, 5, 3, 6, 4, 2, 5, 2, 0, 6)$$

A **Langford sequence of order t and defect d** gives a partition of $\{d, d+1, d+2, \ldots, d+3t-1\}$ into t triples (a_i, b_i, c_i) such that $a_i + b_i = c_i$ for each $i = 1, 2, \ldots, t$.

Similarly, a **hooked Langford sequence of order t and defect d** gives a partition of $\{d, d+1, d+2, \ldots, d+3t-2, d+3t\}$ into t triples (a_i, b_i, c_i) such that $a_i + b_i = c_i$ for each $i = 1, 2, \ldots, t$.

Priday (1959) showed that for every order t, either a Langford or a hooked Langford sequence with defect 2 exists.

Partial results for any defect d were obtained by Davies (1959) and Bermond, Brouwer, and Germa (1978).

The complete solution was given by Simpson in 1983.

Theorem (Simpson, 1983) A Langford sequence of order t and defect d exists if and only if

1. $t \geq 2d - 1$, and

2. $t \equiv 0, 1 \pmod 4$ and d is odd, or $t \equiv 0, 3 \pmod 4$ and d is even.

A hooked Langford sequence of order t and defect d exists if and only if

1. $t(t - 2d + 1) + 2 \geq 0$, and

2. $t \equiv 2, 3 \pmod 4$ and d is odd, or $t \equiv 1, 2 \pmod 4$ and d is even.

In the special case that $d = 2$:

- $\{2, 3, \ldots, 3t + 1\}$ can be partitioned into (a_i, b_i, c_i) with $a_i + b_i = c_i$ for $i = 1, 2, \ldots t$ if and only if $t \geq 3$ and $t \equiv 0, 3 \pmod 4$

- $\{2, 3, \ldots, 3t, 3t + 2\}$ can be partitioned into (a_i, b_i, c_i) with $a_i + b_i = c_i$ for $i = 1, 2, \ldots t$ if and only if $t \geq 1$ and $t \equiv 1, 2 \pmod 4$

Skolem sequences and their many generalizations have applications in numerous areas:

- triple systems, cyclically decomposing complete graphs into 3-cycles (Heffter's Difference Problems)
- starters
- balanced ternary designs
- factorization of complete graphs
- labelings of graphs, including labeling graphs to enhance testing the reliability of a communication network
- generating missile guidance codes resistant to random interference
- design of statistical models, such as a balanced sampling plan excluding contiguous units and a balanced sampling plan avoiding the selection of adjacent units
- Wythoff pairs
- construction of binary sequences with controllable complexity
- testing new parallel processing algorithms

See Nevena Francetić, and Eric Mendelsohn, A Survey of Skolem-type sequences and Rosa's use of them, *Mathematica Slovaca* **59** (2009) 39–76.

Definition For integers m and n with $n \geq m \geq 3$, an m**-cycle system of** K_n is a partition of $E(K_n)$ into m-cycles.

Necessary Conditions: n is odd and $m \mid \frac{n(n-1)}{2}$

Examples

$\boxed{m = 3}$ $\quad 3 \mid \frac{n(n-1)}{2} \Rightarrow n \equiv 1, 3 \pmod 6$, Steiner triple systems, found by Kirkman in 1847

$\boxed{\text{4-cycle system of } K_9}$ where $V(K_9) = \mathbb{Z}_9$

$(0, 1, 8, 5)$
$(1, 2, 0, 6)$
$(2, 3, 1, 7)$
$(3, 4, 2, 8)$
$(4, 5, 3, 0)$
$(5, 6, 4, 1)$
$(6, 7, 5, 2)$
$(7, 8, 6, 3)$
$(8, 0, 7, 4)$

Theorem (Alspach, J., Šajna, 2001-2002) For integers $n \geq m \geq 3$, an m-cycle system of K_n exists if and only if n is odd and $m \mid \frac{n(n-1)}{2}$.

Definition An m-cycle system of K_n where $V(K_n) = \mathbb{Z}_n$ is **cyclic** if for every cycle $C = (v_1, v_2, \ldots, v_m)$ in the m-cycle system, the cycle $C + 1 = (v_1 + 1, v_2 + 1, \ldots, v_m + 1)$ is also in the m-cycle system.

Example

cyclic 4-cycle system of K_9

$(0, 1, 8, 5)$
$(1, 2, 0, 6)$
$(2, 3, 1, 7)$
$(3, 4, 2, 8)$
$(4, 5, 3, 0)$
$(5, 6, 4, 1)$
$(6, 7, 5, 2)$
$(7, 8, 6, 3)$
$(8, 0, 7, 4)$

The necessary conditions for a cyclic m-cycle system are the same as the necessary conditions for an m-cycle system: n is odd and $m \mid \frac{n(n-1)}{2}$.

These necessary conditions are not always sufficient; no cyclic 3-cycle system of K_9 exists.

Definition Let $n \geq 2$ be an integer and let $L \subseteq \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$. The **circulant graph** $\langle L \rangle_n$ is the graph with vertices $V = \mathbb{Z}_n$ and edges $E = \{\{i,j\} : |i-j| \in L$ or $n - |i-j| \in L\}$.

Example $\boxed{\langle 1, 3 \rangle_9}$

Now, $K_n = \langle \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\} \rangle_n$.

To find cyclic 3-cycle systems of K_n, we know $n \equiv 1, 3 \pmod 6$, say $n = 6t + 1$ or $n = 6t + 3$ for some integer t.

Finding cyclic 3-cycle systems of K_n is equivalent to solving ...

Heffter's Difference Problems (1896):

1. Let $n = 6t + 1$. Partition $\{1, 2, \ldots, 3t\}$ into t triples $\{a_i, b_i, c_i\}$ such that $a_i + b_i \equiv \pm c_i \pmod n$.
2. Let $n = 6t + 3$. Partition $\{1, 2, \ldots, 3t + 1\} \setminus \{\frac{n}{3} = 2t + 1\}$ into t triples $\{a_i, b_i, c_i\}$ such that $a_i + b_i \equiv \pm c_i \pmod n$.

Examples

$\boxed{n = 19}$ $\{1, 7, 8\}, \{2, 3, 5\}, \{6, 4, 9\}$

$\boxed{n = 21}$ $\{1, 2, 3\}, \{4, 8, 9\}, \{5, 6, 10\}$

Peltesohn solved both of Heffter's Difference problems:

Theorem (Peltesohn, 1938) For all $n \geq 3$, a cyclic 3-cycle system of K_n exists if and only if $n \equiv 1, 3 \pmod 6$ and $n \neq 9$.

For $t \equiv 0, 1 \pmod 4$, Skolem sequences also solve Heffter's Difference Problem in the case that $n = 6t+1$, and all the triples $\{a, b, c\}$ from the Skolem sequences are of the form $a+b = c$ rather than $a + b \equiv \pm c (\bmod n)$.

For $t \equiv 2, 3 \pmod 4$, hooked Skolem sequences solve Heffter's Difference Problem in the case that $n = 6t + 1$ with all but one of the triples $\{a, b, c\}$ in the form $a+b = c$, namely, the triple that includes $3t + 1$.

For all $t \equiv 0, 1 \pmod 4$, Skolem sequences provide a partition of $\{1, 2, \ldots, 3t\}$ into t triples giving a cyclic 3-cycle system of
$K_{6t+1} = \langle\{1, 2, \ldots, 3t\}\rangle_{6t+1}$.

Example $\boxed{S = (1, 1, 3, 4, 2, 3, 2, 4)}$
$\Rightarrow$ $(1, 5, 6)$, $(2, 9, 11)$, $(3, 7, 10)$, $(4, 8, 12)$
$\Rightarrow$ a cyclic 3-cycle system of $K_{25} = \langle\{1, 2, \ldots, 12\}\rangle_{25}$

For all $t \equiv 2, 3 \pmod 4$, hooked Skolem sequences provide a partition of $\{1, 2, \ldots, 3t - 1, 3t + 1\}$ into t triples also giving a cyclic 3-cycle system of $K_{6t+1} = \langle\{1, 2, \ldots, 3t\}\rangle_{6t+1}$.

Example $\boxed{HS = (1, 1, 2, 0, 2)}$

$\Rightarrow$ $(1, 3, 4)$ and $(2, 5, 7)$

$\Rightarrow$ a cyclic 3-cycle system of $K_{13} = \langle\{1, 2, 3, 4, 5, 6\}\rangle_{13}$

Necessary conditions for a cyclic m-cycle system of K_n: n is odd and $m \mid \frac{n(n-1)}{2}$.

Easiest cases are when (1) $m \mid \frac{n-1}{2}$ so that $n \equiv 1 \pmod{2m}$ or (2) m is odd and $m \mid n$ so that $n \equiv m \pmod{2m}$.

In the case that $n \equiv 1 \pmod{2m}$, say $n = 2mt+1$, so that $K_n = K_{2mt+1} = \langle\{1, 2, \ldots, mt\}\rangle_n$.

Peltesohn (1938): $m = 3$

Kotzig (1965): $m \equiv 0 \pmod 4$

Rosa (1966): $m = 5, 7$
$m \equiv 2 \pmod 4$

El-Zanati, Punnim, Vanden Eynden (2001):
m even

Buratti and Del Fra (2003),
Blinco, El-Zanati, Vanden Eynden (2004),
Fu and Wu (2004): m odd

For m odd and $n \equiv m \pmod{2m}$ we have ...

Peltesohn (1938): $m = 3$ ($n = 9$ was the only exception)

Rosa (1966): $m = 5, 7$

Buratti and Del Fra (2003–2004): m odd with $m \neq 15$ and $m \notin \{p^\alpha \mid p$ is an odd prime and $\alpha \geq 2\}$.

Vietri (2004): for $k \geq 1$, cyclic m-cycle systems of K_{2km+m} exist if $m = 15$ or $m \in \{p^\alpha \mid p$ is an odd prime and $\alpha \geq 2\}$.

Exceptions:

* No cyclic 15-cycle system of K_{15}.

* No cyclic p^α-cycle system of K_{p^α} where p is an odd prime and $\alpha \geq 2$.

Little is known about cyclic m-cycle systems of K_n with $m \mid \frac{n(n-1)}{2}$ and $n \not\equiv 1, m \pmod{2m}$.

Skolem and Langford sequences: partitions of consecutive integers into triples (a, b, c) satisfying $a + b = c$.

So Skolem sequences also give cyclic 3-cycle systems of circulant graphs $\langle\{1, 2, \ldots, 3t\}\rangle_n$ for all $n \geq 6t + 1$ and $t \equiv 0, 1 \pmod 4$.

Similarly, hooked Skolem sequences provide cyclic 3-cycle systems of $\langle\{1, 2, \ldots, 3t - 1, 3t + 1\}\rangle_n$ for $n = 6t + 1$ and all $n \geq 6t + 3$ and $t \equiv 2, 3 \pmod 4$.

Langford sequences provide cyclic 3-cycle systems of $\langle\{d, d+1, d+2, \ldots, d+3t-1\}\rangle_n$ for all $n \geq 6t + 2d - 1$.

Hooked Langford sequences provide cyclic 3-cycle systems of $\langle\{d, d+1, \ldots, d+3t-2, d+3t\}\rangle_n$ for $n = 6t + 2d - 1$ and $n \geq 6t + 2d + 1$.

We wish to extend this partitioning idea to m-tuples $(d_1, d_2, \dots, d_m)$ with $m > 3$ and use these m-tuples to construct cyclic m-cycle systems of circulant graphs.

We will use an equivalent representation with c replaced by $-c$ so that $a + b + c = 0$.

<u>**Definition**</u> An m-tuple $(d_1, d_2, \dots, d_m)$ is of **Skolem-type** if $|d_1|, |d_2|, \dots, |d_m|$ are m distinct positive integers and $d_1 + d_2 + \cdots + d_m = 0$.

A set of t m-tuples $\{(d_{i,1}, d_{i,2}, \dots, d_{i,m}) \mid 1 \leq i \leq t\}$ is called a **Skolem-type m-tuple set of order t** if

- each m-tuple $(d_{i,1}, d_{i,2}, \dots, d_{i,m})$ is of Skolem-type, and
- $\cup_{i=1}^{t} \{|d_{i,1}|, |d_{i,2}|, \dots, |d_{i,m}|\} = \{1, 2, \dots, mt\}$.

A set of t m-tuples $\{(d_{i,1}, d_{i,2}, \ldots, d_{i,m}) \mid 1 \le i \le t\}$ is called a **hooked Skolem-type m-tuple set of order t** if

- each m-tuple $(d_{i,1}, d_{i,2}, \ldots, d_{i,m})$ is of Skolem-type, and
- $\cup_{i=1}^{t}\{|d_{i,1}|, |d_{i,2}|, \ldots, |d_{i,m}|\} = \{1, 2, \ldots, mt-1, mt+1\}.$

Examples

Skolem-type 5-tuple set of order 3

$$\begin{bmatrix} 1 & -2 & 3 & 9 & -11 \\ 4 & -8 & 6 & 13 & -15 \\ 5 & -10 & 7 & 12 & -14 \end{bmatrix}$$

Hooked Skolem-type 6-tuple set of order 3

$$\begin{bmatrix} 1 & -2 & 3 & -4 & -13 & 15 \\ 5 & -6 & 7 & -8 & -17 & 19 \\ 9 & -10 & 11 & -12 & -14 & 16 \end{bmatrix}$$

Theorem Let $m \geq 3$ and $t \geq 1$ be integers.

- A Skolem-type m-tuple set of order t exists if and only if $mt \equiv 0, 3 \pmod 4$.
- A hooked Skolem-type m-tuple set of order t exists if and only if $mt \equiv 1, 2 \pmod 4$.

Theorem Let $m \geq 3$ and $t \geq 1$ be integers.

- For $mt \equiv 0, 3 \pmod 4$ and for all $n \geq 2mt + 1$, a cyclic m-cycle system of $\langle\{1, 2, \ldots, mt\}\rangle_n$ exists.
- For $mt \equiv 1, 2 \pmod 4$ and for $n = 2mt + 1$ and all $n \geq 2mt+3$, a cyclic m-cycle system of $\langle\{1, 2, \ldots, mt-1, mt+1\}\rangle_n$ exists.

We proceed by considering the congruence class of m modulo 4.

The easiest case is when $\boxed{m \equiv 0 \pmod 4}$.

$m \equiv 0 \pmod 4 \Rightarrow mt \equiv 0 \pmod 4$ for all $t \geq 1$

So a Skolem-type m-tuple set of order t exists for all $t \geq 1$.

Example $\boxed{m = 8}$

Each row of the following $t \times 8$ array is a Skolem-type 8-tuple:

$$\boxed{t = 3} \begin{bmatrix} 1 & -2 & -7 & 8 & 13 & -14 & -19 & 20 \\ 3 & -4 & -9 & 10 & 15 & -16 & -21 & 22 \\ 5 & -6 & -11 & 12 & 17 & -18 & -23 & 24 \end{bmatrix}$$

$$\boxed{t = 4} \begin{bmatrix} 1 & -2 & -9 & 10 & 17 & -18 & -25 & 26 \\ 3 & -4 & -11 & 12 & 19 & -20 & -27 & 28 \\ 5 & -6 & -13 & 14 & 21 & -22 & -29 & 30 \\ 7 & -8 & -15 & 16 & 23 & -24 & -31 & 32 \end{bmatrix}$$

In general, for $m \equiv 0 \pmod 4$, say $m = 4r$, a Skolem-type m-tuple set of order t is given by the rows of the following $t \times m$ array A:

$$A = \left[\begin{array}{ccccccc} 1 & -2 & -(2t+1) & 2t+2 & & -[(4r-2)t+1] & (4r-2)t+2 \\ 3 & -4 & -(2t+3) & 2t+4 & & -[(4r-2)t+3] & (4r-2)t+4 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 2t-3 & -(2t-2) & -(4t-3) & 4t-2 & & -(4rt-3) & 4rt-2 \\ 2t-1 & -2t & -(4t-1) & 4t & & -(4rt-1) & 4rt \end{array}\right]$$

$\boxed{m \equiv 2 \pmod 4}$

If t is even, then $mt \equiv 0 \pmod 4$ so we seek a Skolem-type m-tuple set of order t.

If t is odd, then $mt \equiv 2 \pmod 4$ so we seek a hooked Skolem-type m-tuple set of order t.

Example $\boxed{m = 6}$

Each row of the following $t \times 6$ array is a Skolem-type 6-tuple:

$$\boxed{t = 3} \quad \begin{bmatrix} 1 & -2 & 3 & -4 & -13 & 15 \\ 5 & -6 & 7 & -8 & -17 & 19 \\ 9 & -10 & 11 & -12 & -14 & 16 \end{bmatrix}$$

$$\boxed{t = 4} \quad \begin{bmatrix} 1 & -2 & 3 & -4 & -17 & 19 \\ 5 & -6 & 7 & -8 & -21 & 23 \\ 9 & -10 & 11 & -12 & -18 & 20 \\ 13 & -14 & 15 & -16 & -22 & 24 \end{bmatrix}$$

In general, for $m \equiv 2 \pmod 4$:

$t \equiv 0, 2 \pmod 4 \Rightarrow$ the rows of the following $t \times m$ array are Skolem-type m-tuple set of order t.

$$\left[\begin{array}{cccccc|c}
1 & -2 & 3 & -4 & -(4t+1) & 4t+3 & \\
5 & -6 & 7 & -8 & -(4t+5) & 4t+7 & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
2t-3 & -(2t-2) & 2t-1 & -2t & -(6t-3) & 6t-1 & A+6t \\
2t+1 & -(2t+2) & 2t+3 & -(2t+4) & -(4t+2) & 4t+4 & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
4t-3 & -(4t-2) & 4t-1 & -4t & -(6t-2) & 6t &
\end{array}\right]$$

A is the $t \times (m-6t)$ array from the $m \equiv 0 \pmod 4$ case.

$t \equiv 1, 3 \pmod 4 \Rightarrow$ the rows of the following $t \times m$ array are a hooked Skolem-type m-tuple set of order t.

$$\left[\begin{array}{cccccc|c}
1 & -2 & 3 & -4 & -(4t+1) & 4t+3 & \\
5 & -6 & 7 & -8 & -(4t+5) & 4t+7 & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
2t+1 & -(2t+2) & 2t+3 & -(2t+4) & -(6t-1) & 6t+1 & Y+6t \\
2t+5 & -(2t+6) & 2t+6 & -(2t+6) & -(4t+2) & 4t+4 & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
4t-3 & -(4t-2) & 4t-1 & -4t & -(6t-4) & 6t-2 &
\end{array}\right]$$

Y is the $t \times (m - 6t)$ array ...

Y is a $t \times 4r$ array whose entries, in absolute value, are $\{0, 2, 3, \ldots, 4rt-1, 4rt+1\}$.

$$
Y = \left[\begin{array}{cccccccc}
3 & -4 & -(2t+1) & 2t+2 & & & \\
5 & -6 & -(2t+3) & 2t+4 & & & \\
\vdots & \vdots & \vdots & \vdots & \cdots & & \\
2t-3 & -(2t-2) & -(4t-5) & 4t-4 & & & \\
2t-1 & -2t & -(4t-2) & 4t-1 & & & \\
0 & -2 & -(4t-1) & 4t & & &
\end{array}\right.
$$

$$
\left.\begin{array}{ccc}
 & -[(4r-2)t+1] & (4r-2)t+2 \\
 & -[(4r-2)t+3] & (4r-2)t+4 \\
\cdots & \vdots & \vdots \\
 & -(4rt-5) & 4rt-4 \\
 & -(4rt-3) & 4rt-2 \\
 & -(4rt-1) & 4rt+1
\end{array}\right]
$$

$\boxed{m \equiv 3 \pmod 4}$

$\underline{t \equiv 0, 1 \pmod 4}$ A Skolem sequence of order t can be used to find a partition of $\{1, 2, \ldots, 3t\}$ into t triples (a_i, b_i, c_i) with $a_i + b_i = c_i$.

The rows of the following $t \times m$ array are a Skolem-type m-tuple set of order t.

$$\left[\begin{array}{ccc|c} a_1 & -c_1 & b_1 & \\ a_2 & -c_2 & b_2 & \\ & \vdots & & A + 3t \\ a_t & -c_t & b_t & \end{array}\right]$$

A is the $t \times (m - 3t)$ array from the $m \equiv 0 \pmod 4$ case.

$\underline{t \equiv 2,3 \pmod 4}$ A hooked Skolem sequence of order t can be used to find a partition of $\{1,2,\ldots,3t-1,3t+1\}$ into t triples (a_i,b_i,c_i) with $a_i+b_i=c_i$.

The rows of the following $t \times m$ array are a hooked Skolem-type m-tuple set of order t.

$$\left[\begin{array}{ccc|c} a_1 & -c_1 & b_1 & \\ a_2 & -c_2 & b_2 & \\ & \vdots & & Y+3t \\ a_t & -c_t & b_t & \end{array}\right]$$

Y is the $t\times(m-3t)$ array from the $m \equiv 2 \pmod 4$ case.

Finally, the last, and hardest case, $\boxed{m \equiv 1 \pmod 4}$.

For $t \equiv 0, 3 \pmod 4$, we seek a Skolem-type m-tuple of order t; for $t \equiv 1, 2 \pmod 4$ we seek a hooked Skolem-type m-tuple of order t.

Example $\boxed{m = 5}$

$\boxed{t = 4}$ Langford sequence of order 3, defect 2 gives triples

$2 + 7 = 9$

$3 + 5 = 8$

$4 + 6 = 10$

The rows of the following 4×5 array give a Skolem-type 5-tuple set of order 4.

$$\begin{bmatrix} 1 & -2 & 3 & 13 & -15 \\ 4 & -11 & 9 & 17 & -19 \\ 5 & -10 & 7 & 14 & -16 \\ 6 & -12 & 8 & 18 & -20 \end{bmatrix}$$

$\boxed{t=5}$ Langford sequence of order 4, defect 2 gives triples

$2+10=12$

$3+6=9$

$4+7=11$

$5+8=13$

The rows of the following 5×5 array give a hooked Skolem-type 5-tuple set of order 5.

$$\begin{bmatrix} 1 & -2 & 3 & 16 & -18 \\ 4 & -14 & 12 & 20 & -22 \\ 5 & -11 & 8 & 24 & -26 \\ 6 & -13 & 9 & 17 & -19 \\ 7 & -15 & 10 & 21 & -23 \end{bmatrix}$$

In general for the $m \equiv 1 \pmod 4$ case,

- create a $t \times 5$ array:
  - Use a Langford or hooked Langford sequence of order $t-1$, defect 2 to get $t-1$ triples.
  - Add 2 to every integer in each triple and use the triple $\{1, 3, 2\}$. Each triple is "unbalanced" with one side 2 heavier than the other.
  - Pair up consecutive evens. Pairs of consecutive odds. Assign to triples to "balance" them and create 5-tuples.
- Augment with $t \times (m-5t)$ arrays $A+5t$ or $Y+5t$ as needed.

Now that we have the Skolem-type m-tuple sets ... it remains to construct the cycles.

$\boxed{m = 8, t = 3}$

8-cycle system of K_{49} with $V(K_{49}) = \mathbb{Z}_{49}$

$$\begin{bmatrix} 1 & -2 & -7 & 8 & 13 & -14 & -19 & 20 \\ 3 & -4 & -9 & 10 & 15 & -16 & -21 & 22 \\ 5 & -6 & -11 & 12 & 17 & -18 & -23 & 24 \end{bmatrix}$$

Difference 8-tuples:

$(1, -7, 13, -19, -14, 8, -2, 20)$

$(3, -9, 15, -21, -16, 10, -4, 22)$

$(5, -11, 17, -23, -18, 12, -6, 24)$

8-cycles:

$(0, 1, -6, 7, -12, 23, -18, -20)$

$(0, 3, -6, 9, -12, 21, -18, -22)$

$(0, 5, -6, 11, -12, 19, -18, -24)$

$\boxed{m \equiv 0 \pmod 4}$

Suppose we have row i of the matrix A:
$a_{i,1}, a_{i,2}, \ldots, a_{i,m}$.

Recall: $|a_{i,1}| < |a_{i,2}| < \cdots < |a_{i,m}|$, and
$a_{i,j} < 0$ when $j \equiv 2, 3 (\bmod 4)$.

<u>m-tuple:</u> $(a_{i,1}, a_{i,3}, a_{i,5}, \ldots, a_{i,m-3}, a_{i,m-1}, a_{i,m-2},$
$a_{i,m-4}, a_{i,m-6}, \ldots, a_{i,6}, a_{i,4}, a_{i,2}, a_{i,m})$

<u>m-cycle:</u> $(0, a_{i,1}, a_{i,1}+a_{i,3}, a_{i,1}+a_{i,3}+a_{i,5}, \ldots, a_{i,1}+$
$a_{i,3} + a_{i,5} + \cdots + a_{i,m})$

$\boxed{m = 10, t = 3}$

10-cycle system of K_{61} with $V(K_{61}) = \mathbb{Z}_{61}$

$$\begin{bmatrix} 1 & -2 & 3 & -4 & -13 & 15 & 21 & -22 & -25 & 26 \\ 5 & -6 & 7 & -8 & -17 & 19 & 23 & -24 & -27 & 28 \\ 9 & -10 & 11 & -12 & -14 & 16 & 18 & -20 & -29 & 31 \end{bmatrix}$$

Difference 10-tuples:

$(1, -2, 3, -13, 21, -25, -22, 15, -4, 26)$

$(5, -6, 7, -17, 23, -27, -24, 19, -8, 28)$

$(9, -10, 11, -14, 18, -29, -20, 16, -12, 31)$

10-cycles:

$(0, 1, -1, 2, -11, 10, -15, 24, 39, 35)$

$(0, 5, -1, 6, -11, 12, -15, 22, 41, 33)$

$(0, 9, -1, 10, -4, 14, -15, 26, 42, 30)$

$\boxed{m \equiv 2 \pmod 4}$

Suppose we have row i of the matrix:
$a_{i,1}, a_{i,2}, \ldots, a_{i,m}$.

Recall: $|a_{i,1}| < |a_{i,2}| < \cdots < |a_{i,m}|$, and
$a_{i,j} < 0$ when $j = 2$ and $j \equiv 0, 1 \pmod 4$ with
$j \geq 4$.

<u>m-tuple:</u> $(a_{i,1}, a_{i,2}, a_{i,3}, a_{i,5}, \ldots, a_{i,m-3}, a_{i,m-1}, a_{i,m-2}, a_{i,m-4}, a_{i,m-6}, \ldots, a_{i,6}, a_{i,4}, a_{i,m})$

<u>m-cycle:</u> $(0, a_{i,1}, a_{i,1}+a_{i,3}, a_{i,1}+a_{i,3}+a_{i,5}, \ldots, a_{i,1}+a_{i,3}+a_{i,5}+\cdots+a_{i,m})$

Corollary For all $m \geq 3$ and $t \geq 1$, a cyclic m-cycle system of K_{2mt+1} exists.

Corollary For all integers $m \geq 3$ and $t \geq 1$, a cyclic m-cycle system of $K_{2mt+2} - I$ exists if and only if $mt \equiv 0, 3 \pmod{4}$.