

Math 336 – Take-Home Final

Instructions: You must do enough of the following problems to get least 200 points. Each problem must be completed entirely, i.e., you may not select parts (a) and (c) only. Graduate students must do Problems 2, 8, 9, 12, and 19. Your resources for this exam are your class notes, our textbook (and only our textbook), and me. **You are not to discuss this exam with anyone else.** Write your solutions neatly, one to a page, and submit your solutions in order, stapled in the upper left-hand corner.

If your solution to one of the problems below depends on an exercise from the book that was not collected and graded or on another problem below, you must include a proof of that exercise and/or do that problem. You are allowed to use any Theorem/Lemma/Corollary/Definition/Exam 1 & 2 Question/Collected & Graded Exercise from the book without proof.

Your exam must include the following signed statement on the last page:

Signed Statement: With my signature below, I certify that I have only consulted my notes, our book, and Dr. Jordon in arriving at my answers on this exam.

Signed: _____

Due: 10 May, 1:00 PM sharp

Problems:

- (5 pts.) Let G be a group such that $Z(G) \neq \{e\}$ where $Z(G)$ denotes the center of G . Prove that if $H \leq Z(G)$, then $H \trianglelefteq G$.
- (20 pts.) Let G be a group and let $H, K \leq G$.
 - If $H \trianglelefteq G$, show that $HK \leq G$.
 - If $H, K \trianglelefteq G$, show that $HK \trianglelefteq G$.
 - If $H \trianglelefteq G$, show that $H \cap K \trianglelefteq K$.
 - If $H \trianglelefteq G$, show that $K/(H \cap K) \cong HK/H$.
- (5 pts.) Let G be a group and let $H \trianglelefteq G$. If H is cyclic, prove that every subgroup of H is also normal in G .
- (10 pts.) Let G be a group and let $N \trianglelefteq G$. Prove that every subgroup of G/N has the form H/N where H is a subgroup of G .

5. (15 pts.) A group H is **imbedded** in a group G if there exists a one-to-one homomorphism $\phi : H \rightarrow G$. Thus, $\phi(H) \leq G$ and $H \approx \phi(H)$.

(a) Prove that for $n \geq 2$, S_n can be imbedded in A_{n+2} .

(b) Prove that for $n \geq 4$, S_n cannot be imbedded in A_{n+1} .

6. (15 pts.) Let $\sigma \in S_7$ such that $\sigma \neq \epsilon$, the identity.

(a) Enumerate **all** the possible cycle lengths (≥ 2) in the disjoint cycle decomposition of σ . For each possibility, note the sign of σ and the order of σ . Organize your work in a table with the following headings:

Cycle lengths in the disjoint cycle decomposition of σ	sign of σ (even or odd)	order of σ

(b) Using your work in part (a), determine all the possible orders of the elements in S_7 and A_7 .

7. (10 pts.) Let $V = \{\epsilon, (12)(34), (13)(24), (14)(23)\}$.

(a) Prove that $V \leq S_4$.

(b) Determine S_4/V . (**Note:** It is not enough to merely say to which group this group is isomorphic. You must prove whatever you claim.)

8. (10 pts.) Let G be a finite p -group of order p^n . Prove that G contains at least one normal subgroup of order p^i for each $i = 1, 2, \dots, n-1$. (**Hint:** Use induction on i . Recall that $Z(G)$ is nontrivial, use Problem 1, and Cauchy's Theorem from Chapter 24.)

9. (10 pts.) Let G be a finite group with a subgroup H with $|G : H| = p$, where p is the smallest prime divisor of $|G|$. Prove that $H \leq G$.

10. (10 pts.) Let G be a finite group with a normal subgroup H . Let P be a Sylow p -subgroup of G . Prove that $H \cap P$ is a Sylow p -subgroup of H , and that HP/H is a Sylow p -subgroup of G/H . (**Hint:** Compare orders.)

11. (15 pts.)

(a) Let p and q be distinct primes. Prove that every group of order p^2q is not simple.

(b) Prove that a group of order p^nq where p and q are distinct primes with $q < p$ is not simple.

(c) Prove that a group of order pqr , where p, q , and r are not necessarily distinct primes, is not simple.

12. (20 pts.) Prove that there are no simple groups of order n , where $201 \leq n \leq 235$. (Problem 11 might be helpful.)

13. (10 pts.) Consider the groups \mathbb{Q} and \mathbb{Z} under addition.
- Prove that \mathbb{Q}/\mathbb{Z} is an infinite group under addition in which every element has finite order.
 - Show that \mathbb{Q}/\mathbb{Z} has a unique subgroup of order n for every positive integer n .
14. (10 pts.) Let $\text{GL}_2(\mathbb{R})$ denote the set of all 2×2 invertible matrices with entries from \mathbb{R} . Then $\text{GL}_2(\mathbb{R})$ is a group under matrix multiplication. Let $S = \{A \in \text{GL}_2(\mathbb{R}) \mid \det(A) = 1\}$. Let $T = \left\{ \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \mid a \in \mathbb{R}, a \neq 0 \right\}$.
- Show that $\text{GL}_2(\mathbb{R}) = TS$ and $S \trianglelefteq \text{GL}_2(\mathbb{R})$.
 - What conclusion can you draw about the group $\text{GL}_2(\mathbb{R})/S$? (**Hint:** Problem 2(d) might help.)
15. (20 pts.) Let $n \geq 3$. Consider the group A_n , which is a group of permutations the set $S = \{1, 2, \dots, n\}$. Recall the definitions of orbit and stabilizer as given in Chapter 7, pp. 144.
- Determine $\text{orb}_{A_n}(i)$ for each $i \in S$.
 - For $i \in S$, let H_i denote the stabilizer of i , that is, $H_i = \{\sigma \in A_n \mid \sigma(i) = i\}$. Determine $|H_i|$.
 - For $i, j \in \{1, 2, \dots, n\}$, show $H_i \cong H_j$.
 - The group H_n is a group of permutations on $\{1, 2, \dots, n-1\}$ (since an element of H_n fixes n , it must permute the elements $1, 2, \dots, n-1$ among themselves). What is H_n ? (Relate H_n to a familiar group under isomorphism.)
16. (20 pts.) Consider the group S_7 with $5040 = 2^4 \cdot 3^2 \cdot 5 \cdot 7$ elements.
- Exhibit a Sylow 2-subgroup of S_7 (it is sufficient to give a set of generators) and describe the structure of this Sylow 2-subgroup (relate to familiar groups under isomorphism).
 - Exhibit a Sylow 3-subgroup of S_7 (it is sufficient to give a set of generators) and describe the structure of this Sylow 2-subgroup (relate to familiar groups under isomorphism).
 - Exhibit a Sylow 5-subgroup of S_7 (it is sufficient to give a set of generators) and describe the structure of this Sylow 2-subgroup (relate to familiar groups under isomorphism).
 - Exhibit a Sylow 7-subgroup of S_7 (it is sufficient to give a set of generators) and describe the structure of this Sylow 2-subgroup (relate to familiar groups under isomorphism).

17. (10 pts.) In class, we have seen that there are at most three nonisomorphic groups of order 21. In the table on pp. 141 of the text, it appears that there are only 2 nonisomorphic groups of order 21. How can this be? In class, we found the following three groups of order 21:

- $G_1 = \{x^i y^j \mid 0 \leq i \leq 2, 0 \leq j \leq 6, |x| = 3, |y| = 7, xyx^2 = y\}$,
- $G_2 = \{x^i y^j \mid 0 \leq i \leq 2, 0 \leq j \leq 6, |x| = 3, |y| = 7, xyx^2 = y^2\}$, and
- $G_3 = \{x^i y^j \mid 0 \leq i \leq 2, 0 \leq j \leq 6, |x| = 3, |y| = 7, xyx^2 = y^4\}$.

Determine, with explanation, which two of these three groups are isomorphic.

18. (10 pts.)

- Show that there is no homomorphism from $\mathbb{Z}_8 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ onto $\mathbb{Z}_4 \oplus \mathbb{Z}_4$.
- Show that there is no homomorphism from A_4 onto a group of order 2, 4 or 6, but that there is a homomorphism from A_4 onto a group of order 3.

19. (15 pts.) Prove there are exactly four nonisomorphic groups of order 102.