Math 260, Fall 2008
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Answers to the Review Questions for Exam 1

1. Let $A = \{x \in \mathbb{N} \mid x < 7\}$, $B = \{x \in \mathbb{Z} \mid |x-2| < 4\}$, and $C = \{x \in \mathbb{R} \mid x^3 - 4x = 0\}$.
   (a) $A \cup C = \{-2, 0, 1, 2, \ldots, 6\}$, $B \cap C = \{0, 2\}$, $B - C = \{-1, 1, 3, 4, 5\}$, $(A - B) - C = \{6\}$, $A - (B - C) = \{2, 6\}$.
   (b) $S = \{(1,1), (2,0), (3,-1)\}$ and $T = \{(1,2)\}$.

2. (a) False
   (b) True
   (c) False
   (d) False

3. See the answers to Email Questions for 8/25 for some of the answers to these questions.

4. $R = \{(1,1),(1,2),(2,2),(3,3),(4,4),(4,5),(4,6),(5,4),(5,5),(5,6),(6,4),(6,5),(6,6)\}$

5. (a) Not transitive.
   (b) Is an equivalence relation.
   (c) Not symmetric nor transitive.
   (d) Is an equivalence relation.
   (e) Not symmetric.

6. Four positive and five negative integers in [3] are $3, 16, 29, 42, -10, -23, -36$ and in $[-2]$ are $11, 24, 37, 50, -2, -15, -28$.

7. Find the least positive integer a such that
   (a) 4
   (b) 3
   (c) 1
   (d) $10^4 = 4, 10^8 = 2, 10^{12} = 1, 10^{20} = 2, 10^{24} = 1$ in $\mathbb{Z}_7$
   (e) $2, 2^2 = 4, 2^3 = 8, 2^4 = 5, 2^5 = 10, 2^6 = 9, 2^7 = 7, 2^8 = 3, 2^9 = 6, 2^{10} = 1$ in $\mathbb{Z}_{11}$
   (f) $4, 4^2 = 5, 4^3 = 9, 4^4 = 3, 4^5 = 1, 4^6 = 4, 4^7 = 5, 4^8 = 9, 4^9 = 3, 4^{10} = 1$ in $\mathbb{Z}_{11}$

8. (a) $f(n) = n$
   (b) $f(n) = 3n + 1$
   (c) $f(n) = \lceil (n+1)/2 \rceil$
   (d) $f(n) = 5$

9. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 2x + 3$. Suppose $f(x_1) = f(x_2)$ for some $x_1, x_2 \in \mathbb{R}$. Then, $2x_1 + 3 = 2x_2 + 3$ or $2x_1 = 2x_2$ or $x_1 = x_2$ and f is one-to-one. Now, let $y \in \mathbb{R}$. Choose $x = (y-3)/2 \in \mathbb{R}$. Then, $f(x) = f((y-3)/2) = 2(y-3)/2+3 = (y-3)+3 = y$ and thus f is onto.

   (b) $f^{-1}(x) = (x-3)/2$.

   (c) f still one-to-one and but not onto.

10. Inductive steps given below.

   (a) Assume $1 + 2 + 2^2 + 2^3 + \cdots + 2^k = 2^{k+1} - 1$ for some $k \geq 1$. Then

$$\begin{aligned} 1 + 2 + 2^2 + 2^3 + \cdots + 2^{k+1} &= 1 + 2 + 2^2 + 2^3 + \cdots + 2^k + 2^{k+1} \\ &= 2^{k+1} - 1 + 2^{k+1} \\ &= 2 \cdot 2^{k+1} - 1 \\ &= 2^{k+2} - 1. \end{aligned}$$

   (b) Assume $k^2 + k$ is a multiple of 2 for $k \geq 1$, say $k^2 + k = 2\ell$ for some integer ℓ. Then $(k+1)^2 + (k+1) = k^2 + 2k + 1 + k + 1 = k^2 + k + 2k + 2 = 2\ell + 2k + 2 = 2(\ell + k + 1)$. Since $\ell + k + 1$ is an integer, $(k+1)^2 - (k+1)$ is a multiple of 2.

   (c) Assume $1 + 3 + 5 + \cdots + (2k-1) = k^2$ for some $k \geq 1$. Then

$$\begin{aligned} 1 + 3 + 5 + \cdots + (2(k+1) - 1) &= 1 + 3 + 5 + \cdots + (2k-1) + (2k+1) \\ &= k^2 + (2k+1) \\ &= (k+1)^2. \end{aligned}$$

   (d) Assume $2^k > k^2$ for some $k \geq 5$. Then

$$\begin{aligned} 2^{k+1} &= 2 \cdot 2^k \\ &> 2 \cdot k^2 \\ &\geq k^2 + 2k + 1 \text{ since } k \geq 5 \\ &= (k+1)^2. \end{aligned}$$

   (e) Assume $1 \cdot 2 + 3 \cdot 4 + \cdots + (2k-1)(2k) = \dfrac{k(k+1)(4k-1)}{3}$ for some $k \geq 1$. Then

$$\begin{aligned} 1 \cdot 2 + 3 \cdot 4 + \cdots + (2k+1)(2k+2) &= 1 \cdot 2 + 3 \cdot 4 + \cdots + (2k-1)(2k) + (2k+1)(2k+2) \\ &= \frac{k(k+1)(4k-1)}{3} + (2k+1)2(k+1) \\ &= \frac{(k+1)}{3}(k(4k-1) + 6(2k+1)) \\ &= \frac{(k+1)(4k^2 + 11k + 6)}{3} \\ &= \frac{(k+1)(k+2)(4k+3)}{3}. \end{aligned}$$

(f) Assume $(1+\frac{1}{2})^k \geq 1+\frac{k}{2}$ for some $k \geq 1$. Then

$$\begin{aligned}(1+\frac{1}{2})^{k+1} &= (1+\frac{1}{2})^k(1+\frac{1}{2}) \\ &\geq \left(1+\frac{k}{2}\right)(1+\frac{1}{2}) \\ &= 1+\frac{k}{2}+\frac{1}{2}+\frac{k}{4} \\ &\geq 1+\frac{k+1}{2}.\end{aligned}$$

11. (a) Disprove. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 5$ and $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = x$
    (b) Disprove, see (a).
    (c) True, see proofs of problems 65 (in class) and 66 (on homework solutions) of Sec. 2.4.
    (d) True. Let $a_1, a_2 \in A$. Suppose $f(a_1) = f(a_2)$. Then $g(f(a_1)) = g(f(a_2))$ or $(g \circ f)(x) = (g \circ f)(a_2)$. Since $g \circ f$ is one-to-one, we have $a_1 = a_2$.
    (e) Disprove.
    (f) Disprove. Let $f : \mathbb{N} \cup \{0\} \to \mathbb{Z}$ with $f(n) = n$, and let $g : \mathbb{Z} \to \mathbb{N} \cup \{0\}$ with $g(n) = n^2$
    (g) Disprove, see (d).

12. (a) True
    (b) False
    (c) True
    (d) True

13. See the solution to problem 51(a) given in class, Section 3.1.

14. Let a, b, m and n be integers with $m, n \geq 2$ and $m \mid n$. Since $m \mid n$, $n = mk$ for some integer k. Let $a \equiv b \pmod{n}$. Then, $n \mid (a-b)$ so that $a - b = n\ell$ for some $\ell \in \mathbb{Z}$. Thus, $a - b = n\ell = (mk)\ell = m(k\ell)$. Since $k\ell \in \mathbb{Z}$, we have $m \mid (a-b)$ and $a \equiv b \pmod{m}$.

15. (a) See the proof of Theorem 3.2(a).
    (b) Modify the proof of Theorem 3.2(a).
    (c) See the proof of Theorem 3.2(b).

16. Let a, b and $n \geq 2$ be integers.

    (a) Let $a \equiv b \pmod{n}$. Thus, $n \mid (a-b)$ so that $a - b = n\ell$ for some integer ℓ. Let $k \in \mathbb{N}$. Then, $(a-b)k = n\ell k$ or $ak - bk = (nk)\ell$. Since $\ell \in \mathbb{Z}$, we have $nk \mid (ak - bk)$ and thus $ak \equiv bk \pmod{nk}$.
    (b) Disprove: $1 \cdot 2 \equiv 6 \cdot 2 \pmod{10}$ yet $1 \not\equiv 6 \pmod{10}$.