Math 260, Fall 2008
H. Jordon

Review Questions for Exam 1

Throughout this handout, $\mathbb{Z}$ denotes the set of integers, $\mathbb{N}$ denotes the set of positive integers, and $\mathbb{R}$ denotes the set of real numbers.

1. Let $A = \{x \in \mathbb{N} \mid x < 7\}$, $B = \{x \in \mathbb{Z} \mid |x - 2| < 4\}$, and $C = \{x \in \mathbb{R} \mid x^3 - 4x = 0\}$.
   (a) Find each of the following: $A \cup C$, $B \cap C$, $B - C$, $(A - B) - C$, and $A - (B - C)$.
   (b) List the elements in $S = \{(a, b) \in A \times B \mid a + b = 2\}$ and $T = \{(a, c) \in A \times C \mid a \leq c\}$.
2. Let A, B, and C be subsets of some universal set U. For each of the following statements, either prove the given statement is true or exhibit a counterexample to prove it false.
   (a) $A - (B \cup C) = (A - B) \cup (A - C)$
   (b) $(A - B) \times C = (A \times C) - (B \times C)$
   (c) $(A \cup B) \times (C \cup D) = (A \times C) \cup (B \times D)$
   (d) $(A - B) \times (C - D) = (A \times C) - (B \times D)$
3. Let $A = \{1, 2, 3, 4\}$. List the ordered pairs in a relation on A which is
   (a) not reflexive, not symmetric, and not transitive
   (b) reflexive, but neither symmetric nor transitive
   (c) symmetric, but neither reflexive nor transitive
   (d) transitive, but neither reflexive nor symmetric
   (e) reflexive and symmetric, but not transitive
   (f) reflexive and transitive, but not symmetric
   (g) symmetric and transitive, but not reflexive
   (h) reflexive, symmetric, and transitive
4. Given the partition $\mathcal{P}(A) = \{\{1, 2\}, \{3\}, \{4, 5, 6\}\}$ of set A, list the ordered pairs in the equivalence relation R whose equivalence classes are the elements of $\mathcal{P}(A)$.
5. Determine whether or not each of the following relations R defined on the given set A are equivalence relations. If a relation is an equivalence relation, prove it and determine the equivalence class of a specific element; otherwise provide a counterexample to show that it is not an equivalence relation.
   (a) $A = \mathbb{Z}$, $(a, b) \in R$ if $ab \geq 0$
   (b) $A = \mathbb{R}$, $(a, b) \in R$ in $a^2 = b^2$

(c) $A = \mathbb{R}$, $(a, b) \in R$ in $a - b \leq 3$

(d) $A = \mathbb{Z} \times \mathbb{Z}$, $((a, b), (c, d)) \in R$ if $a - c = b - d$

(e) $A = \mathbb{R} \times \mathbb{R}$, $((x, y), (u, v)) \in R$ if $x + y \leq u + v$

6. List four positive and five negative integers in $[3]$ and $[-2]$ in $\mathbb{Z}_{13}$.

7. Find the least positive integer a such that

   (a) $21,758,623 + 17,123,055 \equiv a \pmod{6}$

   (b) $(21,758,623)(17,123,055) \equiv a \pmod{6}$

   (c) $(17,123)^{50} \equiv a \pmod{6}$

   (d) $10^4, 10^8, 10^{12}, 10^{20}, 10^{24}$ modulo 7

   (e) $2, 2^2, 2^3, \ldots, 2^{10}$ modulo 11

   (f) $4, 4^2, 4^3, \ldots, 4^{10}$ modulo 11

8. Give an example of a function $f : \mathbb{N} \to \mathbb{N}$ which is

   (a) one-to-one but not onto

   (b) onto but not one-to-one

   (c) neither one-to-one nor onto

   (d) both one-to-one and onto

9. (a) Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x + 3$ is one-to-one and onto.

   (b) Find the inverse function f^{-1}.

   (c) If $\mathbb{R}$ is replaced with $\mathbb{Z}$ in the definition of f, is f still one-to-one and onto? Justify your answer.

10. Use mathematical induction to establish the truth of each of the following statements for all $n \geq 1$ unless otherwise specified.

    (a) $1 + 2 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 1$

    (b) $1 + 3 + 5 + \cdots + (2n - 1) = n^2$

    (c) $2^n > n^2$, $n \geq 5$

    (d) $1 \cdot 2 + 3 \cdot 4 + \cdots + (2n - 1)(2n) = \dfrac{n(n + 1)(4n - 1)}{3}$

    (e) $(1 + \frac{1}{2})^n \geq 1 + \frac{n}{2}$

11. Prove or disprove the following:

    (a) Let $f : A \to B$ and $g : B \to C$ be two functions. If g is onto, then $g \circ f : A \to C$ is onto.

    (b) Let $f : A \to B$ and $g : B \to C$ be two functions. If g is one-to-one, then $g \circ f : A \to C$ is one-to-one.

(c) If $f : A \to B$ and $g : B \to C$ are both bijective, then $g \circ f : A \to C$ is bijective.

(d) Let $f : A \to B$ and $g : B \to C$ be two functions. If $g \circ f$ is one-to-one, then f is one-to-one.

(e) Let $f : A \to B$ and $g : B \to C$ be two functions. If $g \circ f$ is one-to-one, then g is one-to-one.

(f) There exists functions $f : A \to B$ and $g : B \to C$ such that f is not onto and $g \circ f : A \to C$ is onto.

(g) There exists functions $f : A \to B$ and $g : B \to C$ such that f is not one-to-one and $g \circ f : A \to C$ is one-to-one.

12. Classify each of the following statements as true or false. Explain.

(a) $24 \equiv 3 \pmod 7$

(b) $-17 \equiv 9 \pmod 8$

(c) $-5 \equiv -5 \pmod 4$

(d) $24 \equiv -3 \pmod 3$

13. Let a, b and $n \geq 2$ be integers. Prove that $a \equiv b \pmod n$ if and only if $a = b + kn$ for some integer k.

14. Let a, b, m and n be integers with $m, n \geq 2$ and $m \mid n$. Show that if $a \equiv b \pmod n$, then $a \equiv b \pmod m$.

15. Let a, b, c, d and $n \geq 2$ be integers. Prove each of the following statements assuming $a \equiv b \pmod n$ and $c \equiv d \pmod n$.

(a) $a + c \equiv b + d \pmod n$

(b) $a - c \equiv b - d \pmod n$

(c) $ac \equiv bd \pmod n$

16. Let a, b and $n \geq 2$ be integers.

(a) Prove that if $a \equiv b \pmod n$, then $ak \equiv bk \pmod{nk}$ for every positive integer k.

(b) Prove or disprove: If $ak \equiv bk \pmod n$ for some integer k, then $a \equiv b \pmod n$.