

Extra Homework Problems

Instructions: Answer each of the following questions on a separate piece or pieces of paper. Each question is worth 5 points. You may work with each other to solve the problems but the final write-up must be your own. **Show all work to receive full credit!**

Due: 10 A.M. on Friday, December 5.

1. Let $S = \mathbb{Z} \times \mathbb{Z}$ and define a relation R on S by $(a, b) R (c, d)$ if $a - 2b = c - 2d$. Determine whether or not R is an equivalence relation. If R is an equivalence relation, prove it and determine the equivalence class of $(1, 1)$; otherwise, provide an example which show why R is not an equivalence relation.
2. Let $S = \{a, b, c, d, e, f, g\}$.
 - (a) How many arrangements of distinct letters from S are there of length 5?
 - (b) How many of the arrangements in part (a) contain the letter d ?
 - (c) How many of the arrangements in part (a) contain a d or an e ?
 - (d) How many of the arrangements in part (a) do not contain d or e ?
 - (e) How many of the arrangements in part (a) contain the substring ed ?
3. Use math induction to prove that $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 2 - \frac{1}{n}$ for every positive integer $n \geq 2$. To receive full credit, you must identify the statement $P(n)$, state what you are assuming and what you will prove, and where you used the induction hypothesis. Show all steps to receive full credit.
4. Prove or disprove the following set identities. Recall that to prove a statement, you must show it is true for ALL sets A , B , and C ; to disprove a statement, you need to provide a counterexample.
 - (a) $A - (B - C) = (A - B) \cup (A - \overline{C})$
 - (b) $(A - B) \times (C - D) = (A \times C) - (B \times D)$
5. A function $f : X \rightarrow Y$ has a **left inverse** if there exists a function $\ell : Y \rightarrow X$ such that $(\ell \circ f)(x) = x$ for all $x \in X$. Prove that if $f : X \rightarrow Y$ has a left inverse, then f is one-to-one.
6. A tree T with 36 vertices has 25 vertices of degree 1, two vertices of degree 2, four vertices of degree 4, one vertex of degree 5, two vertices of degree 6, and two vertices of the same (unknown) degree x . What is x ? (Explain your answer.)
7. Let G be a connected graph with $n \geq 3$ vertices and n edges. Prove that G has exactly one cycle.

8. A tournament is **regular** if $\text{outdegree}(v) = \text{indegree}(v)$ for every vertex v .
- (a) Give an example of a tournament of with 7 vertices.
 - (b) Prove that if T is a regular tournament with n vertices, then n is odd.
9. How many 16-bit strings are there containing four 1s and twelve 0s with no consecutive 1s?
10. Twelve basketball players, whose uniforms are numbered 1 through 12, stand around the center ring on the court in an arbitrary arrangement. Show that the sum of the numbers on some group of three consecutive players' uniforms is at least 20.