Name: ______________________

Math 236, Fall 2004
H. Gavlas

Exam 3

104 points possible. 100 points maximum.

1. (16 pts.) Give an example of each of the following or state that no such example exists.

   (a) Monic polynomial of least possible degree in $\mathbb{R}[x]$ with $1-i$ and $3i$ as roots.

   (b) Monic polynomial of least possible degree in $\mathbb{C}[x]$ with $1-i$ and $3i$ as roots.

   (c) An irreducible polynomial of degree at least 3 in $\mathbb{R}[x]$. (Be sure to justify why your polynomial is irreducible if it exists.)

   (d) An irreducible polynomial of degree at least 3 in $\mathbb{Q}[x]$. (Be sure to justify why your polynomial is irreducible if it exists.)

2. (12 pts.) Answer each of the following by CIRCLING True or False. No explanation necessary.

   (a) **True** or **False**: Let F be a field and let $f(x) \in F[x]$. If $f(x)$ is reducible in $F[x]$, then $f(x)$ has a root in F.

   (b) **True** or **False**: Let $n \geq 2$ be an integer. Then $\mathbb{Z}_n[x]$ may have polynomials of positive degree that are units.

   (c) **True** or **False**: Let R be commutative ring with identity and let $f(x), g(x) \in R[x]$. Then $\deg f(x)g(x) \leq \deg f(x) + \deg g(x)$.

   (d) **True** or **False**: Let F be a field and let $f(x) \in F[x]$ be a nonzero polynomial of degree n. Then $f(x)$ has n roots in F.

3. (16 pts.) Find $d(x), u(x)$, and $v(x)$ such that $d(x) = f(x)u(x) + g(x)v(x)$ is the greatest common divisor of $f(x) = 3x^3 + 5x^2 + 6x$ and $g(x) = 4x^4 + 2x^3 + 6x^2 + 4x + 5$ in $\mathbb{Z}_7[x]$.

4. (12 pts.) Show that the following polynomials are irreducible in $\mathbb{Q}[x]$.

(a) $4x^5 + 3x^4 - 12x^2 + 3$

(b) $x^4 + 7x^3 + 14x^2 + 3$

5. (12 pts.) Factor $f(x) = 6x^4 + x^3 + 3x^2 - 14x - 8$ into irreducibles in $\mathbb{Q}[x], \mathbb{R}[x]$, and $\mathbb{C}[x]$.

6. (12 pts.) Let F be a field and let $p(x) \in F[x]$ be a nonzero polynomial with the property that whenever $p(x) \mid r(x)s(x)$ where $r(x), s(x) \in F[x]$, then $p(x) \mid r(x)$ or $p(x) \mid s(x)$. Prove that $p(x)$ is irreducible in $F[x]$ without using Theorem 4.11.

7. (12 pts.) Let F be a field and let $f(x), g(x), h(x) \in F[x]$. Prove that if $f(x) \mid g(x)h(x)$ and $\gcd(f(x), g(x)) = 1_F$, then $f(x) \mid h(x)$.

8. (12 pts.) Let F be field and let

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

and

$$g(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n$$

with $a_i \in F$ for $0 \leq i \leq n$ and $n \geq 1$. Prove that $x - 1_F$ is a factor of $f(x)$ if and only if $x - 1_F$ is a factor of $g(x)$. (**Hint:** Use the fact that 1_F is a root.)