Name: _____

Math 236, Fall 2004
H. Gavlas

Exam 2

105 points possible. 100 points maximum. Throughout this exam, $\mathbb{C}$ denotes the set of complex numbers, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{Z}$ denotes the set of integers, $\mathbb{N}$ denotes the set of positive integers, and $\mathbb{Z}_n$ denotes the set of congruence classes modulo n.

1. (24 pts.) Give an example of each of the following or state that no such example exists.

   (a) A ring without a multiplicative identity.

   (b) A non-commutative ring.

   (c) A commutative ring that is not an integral domain.

   (d) An integral domain that is not a field.

   (e) A finite integral domain that is not a field.

   (f) Two finite non-isomorphic rings with the same number of elements.

   (g) A finite field.

   (h) An infinite field.

2. (15 pts.) Answer each of the following by CIRCLING True or False. No explanation necessary.

   (a) **True** or **False**: If $a, b, c \in R$ where R is a ring and $ab = ac$, then $b = c$.

   (b) **True** or **False**: Let p be prime and let $a \in \mathbb{Z}_p$ with $a \neq 0$. Then the equation $ax = b$ always has a solution for every $b \in \mathbb{Z}_p$.

   (c) **True** or **False**: Let R be a ring with identity. Then the set $U = \{u \in R \mid u \text{ is a unit}\}$ is a subring of R.

   (d) **True** or **False**: Let n be a positive integer such that whenever $ab = 0$ in $\mathbb{Z}_n$ we have $a = 0$ or $b = 0$ in $\mathbb{Z}_n$. Then n is prime.

   (e) **True** or **False**: Let R and S be rings and let $f : R \to S$ be a homomorphism. If R has a multiplicative identity, then $f(1_R) = 1_S$.

3. (8 pts.) Let R be a ring with identity. Prove that if $a \in R$ is a unit, then a is not a zero divisor.

4. (10 pts.) Let R be a ring and let $a, b, c \in R$. Prove that $a(b - c) = ab - ac$.

5. (12 pts.) Find all solutions to $35x = 15$ in $\mathbb{Z}_{60}$.

6. (12 pts.) In class we have seen that $\mathbb{Z}_n$ is a commutative ring with identity. Prove that $\mathbb{Z}_n$ is a field if and only if n is prime.

7. (12 pts.) Let $R = \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$.

(a) Show that R is a subring of $M_2(\mathbb{R})$ by showing that R is closed under subtraction and multiplication.

(b) Is R a commutative ring? Explain.

(c) Does R have a multiplicative identity? If so, what is the identity? If not, why not?

8. (12 pts.) Let $R = \left\{ \begin{bmatrix} a & 0 \\ a & 0 \end{bmatrix} \mid a \in \mathbb{R} \right\}$. Show that R is isomorphic to $\mathbb{R}$ by showing that the function $f : \mathbb{R} \to R$ given by $f(a) = \begin{bmatrix} a & 0 \\ a & 0 \end{bmatrix}$ is a isomorphism.