Name: ____________________

Math 236, Fall 2004
H. Gavlas

Exam 1

105 points possible. 100 points maximum. Throughout this exam, $\mathbb{Z}$ denotes the set of integers, $\mathbb{N}$ denotes the set of positive integers, and $\mathbb{Z}_n$ denotes the set of congruence classes modulo n.

1. (16 pts.) Give an example of each of the following or state that no such example exists.

   (a) Distinct integers a, b, and c such that $a \mid c$ and $b \mid c$ yet $ab \nmid c$.

   (b) Integers a, b and c such that $a \not\equiv 0 \pmod{10}$ and $ab \equiv ac \pmod{10}$ yet $b \not\equiv c \pmod{10}$.

   (c) Positive integers a and b such that $a^2 = 5b^2$.

   (d) Integers a, b and c with $b \neq c$ such that $(a, b) = 1$, $(a, c) = 1$, and $(ab, c) \neq 1$.

2. (10 pts.) Find the prime factorization of 27665. Be sure to list all the numbers you tried as potential factors of 27665, even if they didn't work.

3. (15 pts.) Answer each of the following by CIRCLING True or False. Assume that $a, b, c \in \mathbb{Z} \setminus \{0\}$. No explanation necessary.

   (a) **True** or **False**: If $(a, b) = 1$ and $(a, c) = 1$, then $(a, bc) = 1$.

   (b) **True** or **False**: If $ab \mid c$, then $a \mid c$ and $b \mid c$.

   (c) **True** or **False**: If $a \mid bc$, then $a \mid b$ or $a \mid c$.

   (d) **True** or **False**: If $a \mid (b + c)$, then $a \mid b$ and $a \mid c$.

   (e) **True** or **False**: Let u, v and d be integers such that $au + bv = d$. Then $(a, b) = d$.

4. (8 pts.) Determine the least positive integer a where

$$(98766544321)(823907814364)(6740825628) \equiv a \pmod 9.$$

5. (10 pts.) Prove that 21 divides $4^{n+1} + 5^{2n-1}$ for all positive integers n.

6. (12 pts.) Use the Euclidean Algorithm to find $(8767, 252)$. Use your work to find integers u and v such that $8767u + 252v = (8767, 252)$.

7. (12 pts.) Let $p > 1$ be an integer such that whenever $ab \equiv 0 \pmod{p}$, then $a \equiv 0 \pmod{p}$ or $b \equiv 0 \pmod{p}$. Prove that p is prime.

8. (12 pts.) Let $a, b, n \in \mathbb{Z}$ with $n > 0$.

(a) If $(a, n) = 1$, prove that the equation $ax \equiv 1 \pmod{n}$ has a solution.

(b) If $ab \equiv 1 \pmod{n}$, prove that $(a, n) = (b, n) = 1$.

9. (10 pts.) Find a solution $x \in \mathbb{Z}$ where $0 \leq x < n$ for the following congruences modulo n.

(a) $3x \equiv 7 \pmod{13}$

(b) $8x \equiv 1 \pmod{21}$