Name: ______________________

Math 175, Fall 2007

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Exam 4

105 points possible. 100 points maximum. Throughout this exam, $\mathbb{R}$ denotes the set of real numbers.

1. (15 pts.) Answer each of the following by filling in the blank. No explanation necessary.

   (a) Let A be an invertible $n \times n$ matrix with eigenvalue λ. Then ______ is an eigenvalue of A^{-1}.

   (b) An $n \times n$ matrix A is a diagonalizable if and only if A has n _________ _______________ _______________.

   (c) If S is a set of n nonzero orthogonal vectors in $\mathbb{R}^n$, then S is a _________ for $\mathbb{R}^n$.

   (d) If $\mathbf{v}$ and $\mathbf{w}$ are two vectors in $\mathbb{R}^n$, then the distance between $\mathbf{v}$ and $\mathbf{w}$ is _________.

2. (10 pts.) Let $V = \{(x, y) \mid x, y \in \mathbb{R}\}$ and define the operations of vector addition $\oplus$ and scalar multiplication $\odot$ as follows:

$$(x, y) \oplus (x', y') = (x - y', y - x')$$

   and for a real number c,

$$c \odot (x, y) = (cx, y).$$

   Determine whether scalar multiplication distributes over vector addition, i.e., whether $c \odot (u \oplus v) = c \odot u \oplus c \odot v$ where c is a scalar and u and v are elements of V. Do you think $(V, \oplus, \odot)$ is a vector space? Explain.

3. (20 pts.) Let $A = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & -2 \\ 2 & -2 & 0 \end{bmatrix}$. The eigenvalues of A are 2 and -4.

(a) Determine the matrix whose determinant is the characteristic polynomial for A. (**Note:** You do not have to evaluate the determinant.)

(b) Explain, in a short sentence, how the eigenvalues of the matrix A are related to the determinant of the matrix given in part (a).

(c) Explain, in a short sentence, how the nullspace of the matrix $A + 4I$ consists of the eigenvectors of A for eigenvalue -4.

(d) What is the dimension of the nullspace of the matrix $A + 4I$? Find a basis for the null space of $(A + 4I)$.

(e) What is the dimension of the nullspace of the matrix $A - 2I$? Find a basis for the null space of $(A - 2I)$.

(f) Is A diagonalizable or not? If so, find matrices P and D, where D is a diagonal matrix and $A = PDP^{-1}$. If not, explain why not.

4. (12 pts.) Let W be the subspace of $\mathbb{R}^4$ spanned by $\left\{ \begin{bmatrix} 2 \\ 0 \\ 2 \\ -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -2 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right\}$. Find an orthogonal basis for W.

5. (12 pts.)

(a) Let A be an $m \times n$ matrix and let x be vector in $\mathbb{R}^n$. Prove that $Ax \cdot y = x \cdot A^T y$.

(b) Use part (a) to prove that if A is a symmetric $n \times n$ matrix and λ_1 and λ_2 are distinct eigenvalues with corresponding eigenvectors v_1 and v_2, then v_1 and v_2 are orthogonal.

6. (10 pts.) Prove that if 2 is an eigenvalue of $n \times n$ matrix A, then $2I_n - A$ is not invertible.

7. (12 pts.) Determine, with explanation, whether or not the set W is a subspace of the vector space V.

(a) $W = \left\{ \begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix} \mid a, b \in \mathbb{R} \right\}, V = \mathcal{M}_{2\times 2}$

(b) $W = \{a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 \mid a_0 + a_1 + a_2 + a_3 + a_4 = 0\}, V = \mathcal{P}_4$

8. (14 pts.) For the given set S and vector v, determine if v is in the span of S.

(a) $S = \left\{ \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\}, v = \begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix}$

(b) $S = \{1 + x, x + x^2, x + x^3, 1 + x + x^2 + x^3\}$, $v = 2 - 3x + 4x^2 + x^3$