Name: ____________________

Math 175, Fall 2007
H. Jordon

Exam 3

104 points possible. 100 points maximum. Throughout this exam, $\mathbb{R}$ denotes the set of real numbers.

1. (12 pts.) Answer each of the following by CIRCLING True or False. No explanation necessary.

   (a) **True** or **False**: If T is a linear operator on $\mathbb{R}^n$ and $\mathcal{B}$ is a basis for $\mathbb{R}^n$, then $[T]_{\mathcal{B}}$ is the unique $n \times n$ matrix such that $[T]_{\mathcal{B}}[\mathbf{v}]_{\mathcal{B}} = [T(\mathbf{v})]_{\mathcal{B}}$ for all $\mathbf{v}$ in $\mathbb{R}^n$.

   (b) **True** or **False**: A vector $\mathbf{v}$ is in Col A if and only if $A\mathbf{x} = \mathbf{v}$ is consistent.

   (c) **True** or **False**: Let A be an $m \times n$ matrix. Then $\dim \text{Col } A + \dim \text{Null } A = n$ and $\dim \text{Row } A + \dim \text{Null } A = m$.

   (d) **True** or **False**: The range of a linear transformation is the row space of its standard matrix.

2. (14 pts.) Let $A = \begin{bmatrix} -2 & 2 & 3 & 7 & 1 \\ -2 & 2 & 4 & 8 & 0 \\ -3 & 3 & 2 & 8 & 4 \\ 4 & -2 & 1 & -5 & -7 \end{bmatrix}$ so that $\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

   (a) Determine the rank of A.

   (b) Determine the dimension and find a basis for Col A.

   (c) Determine the dimension and find a basis for Null A.

   (d) Determine the dimension and find a basis for Row A.

3. (21 pts.) Let A be an 5×6 matrix such that $A\mathbf{x} = \mathbf{b}$ has at least one solution for every $\mathbf{b} \in \mathbb{R}^5$. Answer each of the following questions.

(a) The null space of A is a subspace of $\mathbb{R}^k$ for what value of k?

(b) The column space of A is a subspace of $\mathbb{R}^p$ for what value of p?

(c) How many pivots are in the reduced row echelon form of A? How do you know this?

(d) What is the dimension of the null space of A? Why?

(e) What is the dimension of the column space of A? Why?

(f) Is it possible to obtain a basis for $\mathbb{R}^5$ by using some of the columns of A? Why or why not?

(g) Is it possible to obtain a basis for $\mathbb{R}^6$ by using some of the rows of A? Why or why not?

4. (15 pts.) Let A_n denote the following $n \times n$ matrix.

$$A_n = \begin{bmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ -1 & 0 & 3 & 4 & \cdots & n \\ -1 & -2 & 0 & 4 & \cdots & n \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -2 & -3 & -4 & \cdots & 0 \end{bmatrix}$$

(a) Determine A_2, A_3, and A_4. Use elementary row operations and properties of determinants to compute the determinants of these matrices.

(b) Base on your work in part (a), use elementary row operations and properties of determinants to compute $\det A_n$, the determinant of the matrix A_n for an integer $n \geq 2$.

5. (16 pts.) Let $\mathcal{B} = \left\{ \mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \mathbf{b}_3 = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \right\}$ be a basis for $\mathbb{R}^3$. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear operator such that $T(\mathbf{b}_1) = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$, $T(\mathbf{b}_2) = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$, and $T(\mathbf{b}_3) = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$.

(a) Determine $[T(\mathbf{b}_1)]_{\mathcal{B}}$, $[T(\mathbf{b}_2)]_{\mathcal{B}}$, and $[T(\mathbf{b}_3)]_{\mathcal{B}}$.

(b) Determine $[T]_{\mathcal{B}}$, the $\mathcal{B}$-matrix representation of T.

(c) Determine the standard matrix of T.

(d) From your work above, determine an explicit formula for $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right)$.

6. (16 pts.) Let V and W be two subspaces of $\mathbb{R}^n$. Define the set

$$V + W = \{u \in \mathbb{R}^n \mid u = v + w \text{ for some } v \in V \text{ and some } w \in W\}.$$

(a) Prove that $V + W$ is a subspace of $\mathbb{R}^n$.

(b) Let $V = \left\{ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \in \mathbb{R}^4 : v_1 - 2v_2 + v_3 - 3v_4 = 0 \text{ and } -2v_1 + 3v_2 - 3v_3 + 2v_4 = 0 \right\}$ and

$W = \left\{ \begin{bmatrix} -r + 4t \\ r - s + 2t \\ 3t \\ r - t \end{bmatrix} \in \mathbb{R}^4 : r, s, \text{ and } t \text{ are scalars} \right\}$. Find a basis for $V + W$.

7. (10 pts.) Let $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ be a basis for $\mathbb{R}^n$ and let A be an invertible $n \times n$ matrix. Prove that $\{A\mathbf{u}_1, A\mathbf{u}_2, \ldots, A\mathbf{u}_n\}$ is a basis for $\mathbb{R}^n$.