Name: ____________________
Math 175, Fall 2007
H. Jordon

Exam 2

100 points possible. Throughout this exam, $\mathbb{R}$ denotes the set of real numbers.

1. (12 pts.) Answer each of the following by CIRCLING True or False. No explanation necessary.

   (a) **True** or **False**: If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation that is both one-to-one and onto, then $m = n$.

   (b) **True** or **False**: If every column of an $m \times n$ matrix A contains a pivot position, then the matrix equation $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$ in $\mathbb{R}^m$.

   (c) **True** or **False**: If A is an $n \times n$ matrix such that $A^2 = -I_n$, then A is invertible.

   (d) **True** or **False**: If A and B are matrices such that both of the products AB and BA exist, then A and B are square matrices.

2. (9 pts.) Let A be an $m \times n$ matrix such that $A\mathbf{x} = \mathbf{b}$ has at most one solution for each $\mathbf{b}$ in $\mathbb{R}^m$. What three other things (or facts) can you say about A?

3. (9 pts.) Let A be an $n \times n$ matrix such that $A\mathbf{x} = \mathbf{b}$ has exactly one solution for each $\mathbf{b}$ in $\mathbb{R}^n$. What three other things (or facts) can you say about A?

4. (16 pts.) Give an example of each of the following or state that no such example exists.

(a) A 2×2 matrix B such that $BA = 4A$ for every 2×2 matrix A.

(b) A function $f : \mathbb{R}^2 \to \mathbb{R}^2$ that is not a linear transformation.

(c) 2×2 matrices A and B such that A and B are invertible, yet $A + B$ is not invertible.

(d) The 3×3 elementary matrix that corresponds to elementary row operation that adds 2 times the third row to the second row.

5. (10 pts.) Find the largest possible number of linearly independent vectors among

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \quad \mathbf{v}_4 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \quad \mathbf{v}_5 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \quad \mathbf{v}_6 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}.$$

6. (10 pts.) Determine, with explanation, if the following matrices are invertible.

(a) $\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

(b) The $n \times n$ matrix B if $ABC = I_n$ and A and C are $n \times n$ invertible matrices.

7. (12 pts.) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 + 3x_3 \\ x_1 + x_2 + 4x_3 \\ 2x_1 + 4x_2 + x_3 \end{bmatrix}.$$

(a) Determine the standard matrix of T.

(b) Determine, with explanation, if T is onto.

(c) Determine, with explanation, if T is one-to-one.

8. (10 pts.) Let A be a 6×5 matrix such that the nullity of A is 0. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be the linear transformation defined by $T(\mathbf{x}) = A\mathbf{x}$. Answer each of the following questions. **Be sure to justify your answers.**

   (a) What is the value of m? What is the value of n?

   (b) What is the maximum number of linearly independent vectors in the range of T?

   (c) Is T onto?

   (d) Is T one-to-one?

9. (12 pts.) Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in $\mathbb{R}^n$. Prove that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent if and only if the set $\{\mathbf{u}, \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{w}\}$ is linearly independent.