

**Exam 1**

106 points possible. 100 points maximum. Throughout this exam,  $\mathbb{R}$  denotes the set of real numbers.

1. (18 pts.) Answer each of the following by CIRCLING True or False. No explanation necessary.

- (a) **True** or **False**: If  $\mathbf{v}$  is a vector in  $\mathbb{R}^n$ , then any scalar multiple of  $\mathbf{v}$  is a linear combination of  $\mathbf{v}$ .
- (b) **True** or **False**: If  $A$  is an  $m \times n$  matrix with rank  $m$ , then the span of the columns of  $A$  is  $\mathbb{R}^m$ .
- (c) **True** or **False**: If a system of linear equations is consistent, then the reduced row echelon form of the augmented matrix for this system contains a zero row.
- (d) **True** or **False**: If  $A$  is a matrix with rank  $k$ , then the vectors  $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_k$  appear as columns of the reduced row echelon form of  $A$ .
- (e) **True** or **False**: Let  $\mathcal{S}$  be a nonempty set of vectors in  $\mathbb{R}^n$ , and let  $\mathbf{v}$  be a vector in  $\mathbb{R}^n$ . Then  $\text{Span } \mathcal{S} = \text{Span } (\mathcal{S} \cup \{\mathbf{v}\})$  if and only if  $\mathbf{v}$  is in the span of  $\mathcal{S}$ .
- (f) **True** or **False**: If  $A$  is an  $m \times n$  matrix, then the only vector  $\mathbf{u}$  with  $A\mathbf{u} = \mathbf{0}$  is  $\mathbf{u} = \mathbf{0}$ .

2. (12 pts.) Let  $A$  be an  $m \times n$  matrix, and let  $\mathbf{u}$  and  $\mathbf{v}$  be  $n \times 1$  vectors. Without using Theorem 1.3, prove that  $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$ .

3. (24 pts.) Give an example of each of the following or state that no such example exists.

(a)  $2 \times 2$  matrix  $A$  and  $2 \times 1$  nonzero vectors  $\mathbf{u}$  and  $\mathbf{v}$  such that  $A\mathbf{u} = A\mathbf{v}$  yet  $\mathbf{u} \neq \mathbf{v}$ .

(b) A  $2 \times 3$  matrix  $A$  such that the equation  $A\mathbf{x} = \mathbf{b}$  has at least one solution for each  $\mathbf{b} \in \mathbb{R}^2$ .

(c) A set  $S$  containing two vectors from  $\mathbb{R}^3$  such that  $\text{Span } S = \mathbb{R}^3$ .

(d) A  $3 \times 4$  augmented matrix in reduced row echelon form so that the corresponding system of equations has infinitely many solutions.

(e) A  $3 \times 4$  augmented matrix in reduced row echelon form so that the corresponding system of equations has exactly one solution.

(f) A  $3 \times 4$  augmented matrix in reduced row echelon form so that the corresponding system of equations has no solution.

4. (20 pts.) Find the general solution in vector form to

$$\begin{aligned}x_1 - x_2 - 2x_3 - 8x_4 &= -3 \\-2x_1 + x_2 + 2x_3 + 9x_4 &= 5 \\3x_1 - 2x_2 - 3x_3 - 15x_4 &= -9\end{aligned}$$

using the Gaussian elimination algorithm. (**Note:** This means you are allowed to use your calculator only as a check.)

5. (10 pts.) Let  $A$  be an  $m \times n$  matrix, let  $\mathbf{b}$  be a vector in  $\mathbb{R}^m$ , and suppose that  $\mathbf{v}$  is a solution of  $A\mathbf{x} = \mathbf{b}$ .

(a) Prove that if  $\mathbf{w}$  is a solution of  $A\mathbf{x} = \mathbf{0}$ , then  $\mathbf{v} + \mathbf{w}$  is a solution of  $A\mathbf{x} = \mathbf{b}$ .

(b) Prove that for any solution  $\mathbf{u}$  to  $A\mathbf{x} = \mathbf{b}$ , there is a solution  $\mathbf{w}$  to  $A\mathbf{x} = \mathbf{0}$ .

6. (10 pts.) Determine, if possible, a value of  $r$  for which  $\mathbf{v} = \begin{bmatrix} -1 \\ 3 \\ r \end{bmatrix}$  is in the span of

$$\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

7. (12 pts.) Consider an economy that is divided into three sectors: agriculture, manufacturing, and services. Suppose that each dollar's worth of agricultural output requires inputs of \$0.10 from the agricultural sector, \$0.15 from the manufacturing sector, \$0.30 from the services sector; each dollar's worth of manufacturing output requires inputs of \$0.20 from the agricultural sector, \$0.25 from the manufacturing sector, \$0.10 from the services sector; each dollar's worth of services output requires inputs of \$0.20 from the agricultural sector, \$0.35 from the manufacturing sector, \$0.10 from the services sector. (**Note:** You are expected to use a calculator to help you solve this problem.)

(a) What is the net production corresponding to a gross production of \$40 million of agriculture, \$50 million of manufacturing, and \$30 million of services?

(b) What gross production is required to satisfy exactly a demand for \$90 million of agriculture, \$72 million of manufacturing, and \$96 million of services?