Topic: u Substitution

One method for taking integrals $\int f(x)\,dx$ is called u substitution. u substitution requires identifying a function $u(x)$ such that the integral $\int g(u)\,du$ is simpler than the original integral $\int f(x)\,dx$, where the function $g(u)$ comes from replacing occurrences of $u(x)$ inside the function $f(x)$ by the new variable u, and du comes from the equation $du = u'(x)dx$.

After making a choice of the function $u(x)$, one must decide where in the function $f(x)$ the replacements will be made. Then, dx should be replaced by $\frac{du}{u'(x)}$. The goal is that after both these steps, there are only us in the integral and no xs. If this condition is met, and the new integral is simpler than the old one, then the substitution was successful, and it may be posible to find the new integral directly.

In the case of definite integrals, the endpoints of integration need to be adjusted when applying u substitution. If the endpoints were real numbers a and b in the original integral, then they should be $u(a)$ and $u(b)$ in the transformed integral.

Examples:

1. Find the integral $\int xe^{x^2}\,dx$.

We set $u = x^2$, and then $du = 2x\,dx$, or $dx = \frac{du}{2x}$. We replace the x^2 in the exponent by u and also replace dx by its expression in terms of du to obtain

$$\int xe^{x^2}\,dx = \int xe^u\,\frac{du}{2x} = \frac{1}{2}\int e^u du$$

Since the integral on the far right does not contain any x and is much simpler than the original, we can take the integral. $\frac{1}{2}\int e^u du = \frac{1}{2}e^u + C$.

2. Find the value of the definite integral $\int_{\pi/3}^{\pi/2} \sin^5 x \cos x\,dx$.

We set $u = \sin x$, so $du = \cos x dx$ or $dx = \frac{du}{\cos x}$. We replace $\sin^5 x$ by u^5 and dx by $\frac{du}{\cos x}$. This transofrms ths integrand into $u^5 \cos x \frac{du}{\cos x} = u^5\,du$, which is simpler than the original and has no xs. Since this is a definite integral, we must also apply u to both of the endpoints. The transformed integral becomes

$$\int_{\sin \pi/3}^{\sin \pi/2} u^5\,du = \int_{\sqrt{3}/2}^{1} u^5\,du = \frac{1}{6}u^6\bigg|_{\sqrt{3}/2}^{1} = \frac{1}{6}(1^6 - (\sqrt{3}/2)^6) = \frac{1}{6}(1 - \frac{27}{64}) = \frac{37}{384}$$

3. Find the integral $\int \frac{x+1}{\sqrt{x-1}}\,dx$.

We set $u = x - 1$, and note that $x = u + 1$. Then $du = dx$. We replace $x - 1$ in the denominator by u, and $x + 1$ in the numerator by $u + 2$. Since $du = dx$, the integral becomes $\int \frac{u+2}{\sqrt{u}}\, du$. Now, we write $\frac{1}{\sqrt{u}}$ as $u^{-1/2}$ to obtain

$$\int (u+2)u^{-1/2}\, du = \int u^{1/2} + 2u^{-1/2}\, du = \frac{u^{3/2}}{3/2} + 2\frac{u^{1/2}}{1/2} + C = \frac{2}{3}u^{3/2} + 4u^{1/2} + C$$

4. Find the value of the definite integral $\int_2^3 x^3(x^2+1)^{1/8}\, dx$.

We set $u = x^2 + 1$, so $du = 2x\, dx$. We replace the $x^2 + 1$ by u, and we also replace an x^2 (coming from the x^3 factor) with $u - 1$. The integral then becomes $\int_2^3 (u-1)u^{1/8} x dx$. Noting that $dx = \frac{du}{2x}$, we replace dx in the integral. We must also change the bounds of integration, so the 2 becomes $2^2 + 1 = 5$ and the 3 becomes $3^2 + 1 = 10$. We get

$$\int_5^{10} (u-1)u^{1/8}\frac{1}{2}du = \frac{1}{2}\int_5^{10} u^{9/8} - u^{1/8}\, du = \frac{1}{2}\left(\frac{u^{17/8}}{17/8} - \frac{u^{9/8}}{9/8}\right)\Bigg|_5^{10}$$

$$= \frac{1}{2}\left(\frac{8}{17}(10^{17/8} - 5^{17/8}) - \frac{8}{9}(10^{9/8} - 5^{9/8})\right)$$