# Topic: Product and Quotient Rule

**Background:**

**Product Rule:** $[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$

**Quotient rule:** $\left(\dfrac{f(x)}{g(x)}\right)' = \dfrac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$

**Illustrative Examples:**

(1) Find the derivative of the function $h(x)$ given below.

$$h(x) = (4x^5 - 3x^2 + 7x + 1)(x^3 + 3x + 7)$$

Solution:

$h(x) = f(x)g(x)$ where $f(x) = (4x^5 - 3x^2 + 7x + 1)$ and $g(x) = (x^3 + 3x + 7)$.

$f'(x) = (4)(5)x^{(5-1)} - (3)(2)x^{(2-1)} + 7 = 20x^4 - 6x + 7$

$g'(x) = 3x^{(3-1)} + 3 = 3x^2 + 3$

Hence,

$$\begin{aligned} h'(x) &= f'(x)g(x) + f(x)g'(x) \\ &= (20x^4 - 6x + 7)(x^3 + 3x + 7) + (4x^5 - 3x^2 + 7x + 1)(3x^2 + 3) \end{aligned}$$

(2) Find the derivative of the function $r(t)$ given below.

$$r(t) = 2t^3 e^{5t}$$

Solution:

$r(t) = f(t)g(t)$ where $f(t) = 2t^3$ and $g(t) = e^{5t}$.

$f'(t) = (2)(3)t^{(3-1)} = 6t^2$

$g'(t) = 5e^{5t}$

Hence,

$$\begin{aligned} r'(t) &= f'(t)g(t) + f(t)g'(t) \\ &= (6t^2)(e^{5t}) + (2t^3)(5e^{5t}) \\ &= 6t^2e^{5t} + 10t^3e^{5t} \end{aligned}$$

(3) Consider the function $f(y) = e^{4y} g(y)$, where $g(0) = 2$ and $g'(0) = 5$. Find $f'(0)$.

Solution:

$f(y) = h(y)g(y)$ where $h(y) = e^{4y}$.

$h'(y) = 4e^{4y}$

$f'(y) = h'(y)g(y) + h(y)g'(y) = 4e^{4y}g(y) + e^{4y}g'(y)$

Hence,

$$\begin{aligned} f'(0) &= 4e^{(4)(0)}g(0) + e^{(4)(0)}g'(0) \\ &= (4)(2) + (1)(5) \\ &= 13 \end{aligned}$$

(4) Find the derivative of the function $h(x)$ given below.

$$h(x) = \frac{x^3 - 4x}{2^x + 3x}$$

Solution:

$h(x) = \frac{f(x)}{g(x)}$ where $f(x) = x^3 - 4x$ and $g(x) = 2^x + 3x$.

$f'(x) = 3x^2 - 4$

$g'(x) = 2^x \ln(2) + 3$

Hence,

$$\begin{aligned} h'(x) &= \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2} \\ &= \frac{(3x^2 - 4)(2^x + 3x) - (x^3 - 4x)(2^x \ln(2) + 3)}{(2^x + 3x)^2} \end{aligned}$$

(5) Consider the function $f(x) = \frac{h(x)k(x)}{g(x)}$. Suppose that $h(1) = 1, k(1) = 2, g(1) = 3, h'(1) = 2, k'(1) = 4, g'(1) = 5$. Find $f'(1)$.

Solution:

$$\begin{aligned} f'(x) &= \frac{[h(x)k(x)]g'(x) - [h(x)k(x)]'g(x)}{(g(x))^2} \quad \text{(using Quotient Rule)} \\ &= \frac{[h(x)k(x)]g'(x) - [h'(x)k(x) + h(x)k'(x)]g(x)}{(g(x))^2} \\ &\quad \text{(using Product Rule on } h(x)k(x)) \end{aligned}$$

Hence,

$$\begin{aligned} f'(1) &= \frac{[h(1)k(1)]g'(1) - [h'(1)k(1) + h(1)k'(1)]g(1)}{[g(1)]^2} \\ &= \frac{[(1)(2)](5) - [(2)(2) + (1)(4)](3)}{(3)^2} \\ &= -\frac{14}{9} \end{aligned}$$

(6) Consider the function $f(x) = x^3 \sin(2x)$. Find $f'(\frac{\pi}{2})$.

Solution:

$f(x) = h(x)g(x)$ where $h(x) = x^3$ and $g(x) = \sin(2x)$.

$h'(x) = 3x^2$

$g'(x) = 2\cos(2x)$

Hence, $f'(x) = 3x^2 \sin(2x) + 2x^3 \cos(2x)$, and,

$$\begin{aligned} f'(\frac{\pi}{2}) &= 3\frac{\pi^2}{4}\sin(\pi) + 2\frac{\pi^3}{8}\cos(\pi) \\ &= -\frac{\pi^3}{4} \end{aligned}$$

**Cautions:**

$$\boxed{[f(x)g(x)]' \neq f'(x)g'(x)}$$

$$\boxed{\left(\frac{f(x)}{g(x)}\right)' \neq \frac{f'(x)}{g'(x)}}$$