Topic: Logarithms

Definition: The logarithm base b of x is denoted by $\log_b x$, where both b and x should be positive. The expression $\log_b x$ asks, to what power must I raise b in order to get x? In other words, if $x = b^n$, then $\log_b x = n$, because x is b to the nth power.

Logarithms are the inverse functions of exponentials. In particular, $\boxed{\log_b b^x = x}$ and $\boxed{b^{\log_b x} = x}$.

When there is a product inside a log, we can split the log into the sum of two logs: $\boxed{\log_b xy = \log_b x + \log_b y}$

When there is a quotient inside a log, we can split the log into the difference of two logs: $\boxed{\log_b \frac{x}{y} = \log_b x - \log_b y}$

When there is an exponent inside a log, the exponent can come outside as a multiplier: $\boxed{\log_b x^n = n \log_b x}$

Note: everything inside the log must be raised to the same exponent in order to pull the exponent out. For example, $\log_b x^n y \neq n \log_b xy$, but $\log_b x^n y^n = \log_b (xy)^n = n \log_b xy$.

Lastly, we have the change of base formula: $\boxed{\log_b x = \frac{\log_c x}{\log_c b}}$ for any intermediate base c.

Another nice property of logs is that $\log_x y = \frac{1}{\log_y x}$.

Examples

1. Find x such that $8^x = \frac{1}{4}$.

The answer to this question is equivalent to calculating $\log_8 \frac{1}{4}$. By the change of base formula, this is $\frac{\log 1/4}{\log 8}$. Now, we write everything in powers of two, to obtain $\frac{\log 2^{-2}}{\log 2^3}$. By the property for exponents inside of logs, we obtain $\frac{-2 \log 2}{3 \log 2} = \frac{-2}{3}$.

2. For the function $f(x) = (\frac{1}{2})^x$, calculate $f(4)$.

We simply plug in 4 for x, to obtain $f(4) = (\frac{1}{2})^4 = \frac{1}{2^4} = \frac{1}{16}$.

3. Find the exponential function $f(x) = a^x$ which passes through the point $(3, 64)$.

This problem is asking us to find the value of a which makes the function pass through the given point. Substituting the given values gives $64 = a^3$. Then $a = \sqrt[3]{64} = 4$.

4. Use the properties of logarithms to rewrite the expression $\log \left(\frac{x^4 y^5}{z^7} \right)$ in the form $A \log x + B \log y +$

$C \log z$.

We first apply the properties for products and quotients inside of logs to obtain $\log x^4 + \log y^5 - \log z^7$. Then we apply the property for an exponent inside a log to obtain $4 \log x + 5 \log y - 7 \log z$. Therefore, $A = 4, B = 5, C = -7$.

5. Rewrite the expression $\log 2 + 3 \log x - 4 \log(x - 1)$ as a single logarithm.

This is like the previous problem, but in reverse. First we change $3 \log x$ into $\log x^3$ and $4 \log(x - 1)$ into $\log(x - 1)^4$. The addition of logs becomes the log of a product, so we get $\log 2x^3 - \log(x - 1)^4$, and the difference of logs becomes the log of a quotient, so we obtain $\log \frac{2x^3}{(x-1)^4}$.

6. Evaluate $\log_4 .0625$.

Since 4 is a power of 2, we aim to write .0625 as a power of 2 as well. By writing $.0625 = \frac{625}{10000}$ and reducing the fraction, we find that $.0625 = \frac{1}{16}$. By the change of base formula, the desired log is $\frac{\log 1/16}{\log 4} = \frac{\log 2^{-4}}{\log 2^2} = \frac{-4 \log 2}{2 \log 2} = \frac{-4}{2} = -2$

7. Evaluate $\log 4 + \log 25$.

The efficient way to do this problem is to combine the logs, rather than trying to calculate the values separately and then adding them. When we combine the logs using the sum/product property, we obtain $\log 4 \cdot 25 = \log 100 = \log 10^2$. Now, the log and the exponential cancel (since log means the logarithm base 10) so the answer is 2.

8. Evaluate $9^{\log_3 2}$.

We want the log and the exponential to cancel, but can't cancel them yet because the base of the exponential is not the same as the base of the log. So we write $9 = 3^2$ and use properties of exponents to rearrange: $9^{\log_3 2} = (3^2)^{\log_3 2} = 3^{2 \log 3_2} = (3^{\log_3 2})^2 = 2^2 = 4$. The logs do cancel in the second to last step because the bases are the same.