

RAMANUJAN'S SERIES FOR $1/\pi$: A SURVEY

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To

Charles Vanden Eynden

on the occasion of his

$$\left\{ \frac{722204136308737}{53747712} + \sqrt{\frac{521578814501445883951237300225}{2888816545234944}} \right\}^{1/4}$$

birthday

Abstract. Beginning with Ramanujan's epic paper, *Modular Equations and Approximations to π* , we describe Ramanujan's series for $1/\pi$ and later attempts to prove them. Generalizations, analogues, and consequences of Ramanujan's series are discussed.

1. THE BEGINNING

Toward the end of the first paper [55], [56, p. 36] that Ramanujan published in England, at the beginning of Section 13, he writes, "I shall conclude this paper by giving a few series for $1/\pi$ ". (In fact, Ramanujan concluded his paper a couple pages later with another topic: formulas and approximations for the perimeter of an ellipse.) After sketching his ideas, which we examine in detail in Sections 3 and 9, Ramanujan records three series representations for $1/\pi$. As customary, set

$$(a)_0 := 1, \quad (a)_n := a(a+1) \cdots (a+n-1), \quad n \geq 1.$$

Let

$$A_n := \frac{\left(\frac{1}{2}\right)_n^3}{n!^3}, \quad n \geq 0. \quad (1.1)$$

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Theorem 1.1. *If A_n is defined by (1.1), then*

$$\frac{4}{\pi} = \sum_{n=0}^{\infty} (6n+1) A_n \frac{1}{4^n}, \quad (1.2)$$

$$\frac{16}{\pi} = \sum_{n=0}^{\infty} (42n+5) A_n \frac{1}{2^{6n}}, \quad (1.3)$$

$$\frac{32}{\pi} = \sum_{n=0}^{\infty} \left((42\sqrt{5}+30)n + 5\sqrt{5}-1 \right) A_n \frac{1}{2^{6n}} \left(\frac{\sqrt{5}-1}{2} \right)^{8n}. \quad (1.4)$$

The first two formulas, (1.2) and (1.3), appeared in the Walt Disney film, “High School Musical,” starring Vanessa Anne Hudgens, who plays an exceptionally bright high school student named Gabriella Montez. Gabriella points out to her teacher that she had incorrectly written the left-hand side of (1.3) as $8/\pi$ instead of $16/\pi$ on the blackboard. After first claiming that Gabriella is wrong, her teacher checks (possibly Ramanujan’s *Collected Papers*?) and admits that Gabriella is correct. Formula (1.2) was correctly recorded on the blackboard.

After offering the three formulas for $1/\pi$ given above, at the beginning of Section 14 [55], [56, p. 37], Ramanujan claims, “There are corresponding theories in which q is replaced by one or other of the functions”

$$q_r := q_r(x) := \exp \left(-\pi \csc(\pi/r) \frac{{}_2F_1\left(\frac{1}{r}, \frac{r-1}{r}; 1; 1-x\right)}{{}_2F_1\left(\frac{1}{r}, \frac{r-1}{r}; 1; x\right)} \right), \quad (1.5)$$

where $r = 3, 4$, or 6 , and where ${}_2F_1$ denotes one of the hypergeometric functions ${}_pF_{p-1}$, $p \geq 1$, which are defined by

$${}_pF_{p-1}(a_1, \dots, a_p; b_1, \dots, b_{p-1}; x) := \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_{p-1})_n} \frac{x^n}{n!}, \quad |x| < 1.$$

Ramanujan then offers 14 further series representations for $1/\pi$. Of these, 10 belong to the quartic theory, i.e., for $r = 4$; 2 belong to the cubic theory, i.e., for $r = 3$; and 2 belong to the sextic theory, i.e., for $r = 6$. Ramanujan never returned to the “corresponding theories” in his published papers, but six pages in his second notebook [57] are devoted to developing these theories, with all of the results on these six pages being proved in a paper [15] by Berndt, S. Bhargava, and F. G. Garvan. That the classical hypergeometric function ${}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right)$ in the classical theory of elliptic functions could be replaced by one of the three hypergeometric functions above and concomitant theories developed is one of the many incredibly ingenious and useful ideas bequeathed to us by Ramanujan. The development of these theories is far from easy and is an active area of contemporary research.

All 17 series for $1/\pi$ were discovered by Ramanujan in India before arriving in England, for they can be found in his notebooks [57], which were written prior to Ramanujan’s departure for England. In particular, (1.2), (1.3), and (1.4) can be found on page 355 in his second notebook and the remaining 14 series are found in his third notebook [57, p. 378]; see also [13, pp. 352–354]. It is interesting that (1.2), (1.3), and (1.4) are

also located on a page published with Ramanujan's lost notebook [58, p. 370]; see also [3, Chapter 15].

2. THE MAIN ACTORS FOLLOWING IN THE FOOTSTEPS OF RAMANUJAN

Fourteen years after the publication of [55], the first mathematician to address Ramanujan's formulas was Sarvadaman Chowla [35], [36], [37, pp. 87–91, 116–119], who gave the first published proof of a general series representation for $1/\pi$ and used it to derive the first (1.2) of Ramanujan's series for $1/\pi$ [55, Eq. (28)]. We briefly discuss Chowla's ideas in Section 4.

Ramanujan's series were then forgotten by the mathematical community until November, 1985, when R. William Gosper, Jr. used one of Ramanujan's series, namely,

$$\frac{9801}{\pi\sqrt{8}} = \sum_{n=0}^{\infty} \frac{(4n)!(1103 + 26390n)}{(n!)^4 396^{4n}}, \quad (2.1)$$

to calculate 17,526,100 digits of π , which at that time was a world record. There was only one problem with his calculation – (2.1) had not yet been proved. However, a comparison of Gosper's calculation of the digits of π with the previous world record held by Y. Kanada made it extremely unlikely that (2.1) was incorrect.

In 1987, Jonathan and Peter Borwein [22] succeeded in proving all 17 of Ramanujan's series for $1/\pi$. In a subsequent series of papers [23], [24], [28], they established several further series for $1/\pi$, with one of their series [28] yielding roughly fifty digits of π per term. The Borweins were also keen on calculating the digits of π , and accounts of their work can be found in [29], [27], and [25].

At about the same time as the Borweins were devising their proofs, David and Gregory Chudnovsky [38] also derived series representations for $1/\pi$ and, in particular, used their series

$$\frac{1}{\pi} = 12 \sum_{n=0}^{\infty} (-1)^n \frac{(6n)!}{(n!)^3 (3n)!} \frac{13591409 + 545140134n}{(640320)^{3n+3/2}} \quad (2.2)$$

to calculate a world record 2,260,331,336 digits of π . The series (2.2) yields 14 digits of π per term. A popular account of the Chudnovskys' calculations can be found in a paper written for *The New Yorker* [54].

The third author of the present paper and his coauthors (Berndt, S. H. Chan, A. Gee, W.-C. Liaw, Z.-G. Liu, V. Tan, and H. Verrill) in a series of papers [18], [30], [31], [32], [34] extended the ideas of the Borweins, in particular, without using Clausen's formula in [30] and [34], and derived general hypergeometric-like formulas for $1/\pi$. We devote Section 8 of our survey to discussing some of their results.

Stimulated by the work and suggestions of the third author, the first two authors [7], [8] systematically returned to Ramanujan's development in [55] and employed his ideas in order not only to prove most of Ramanujan's original representations for $1/\pi$ but also to establish a plethora of new such identities as well. In another paper [9], motivated by the work of Jesús Guillera [46]–[51], who both experimentally and rigorously discovered many new series for both $1/\pi$ and $1/\pi^2$, the first two authors continued to follow Ramanujan's ideas and devised series representations for $1/\pi^2$.

In the survey which follows, we delineate the main ideas in Sections 3, 6, 7, 8, and 9, where the ideas of Ramanujan, the Borwein brothers, the Chudnovsky brothers, Chan and his coauthors, and the present authors, respectively, are discussed.

3. RAMANUJAN'S IDEAS

To describe Ramanujan's ideas, we need several definitions from the classical theory of elliptic functions, which, in fact, we use throughout the paper. The complete elliptic integral of the first kind is defined by

$$K := K(k) := \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}}, \quad (3.1)$$

where $k, 0 < k < 1$, denotes the *modulus*. Furthermore, $K' := K(k')$, where $k' := \sqrt{1 - k^2}$ is the *complementary modulus*. The complete elliptic integral of the second kind is defined by

$$E := E(k) := \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \varphi} d\varphi.$$

If $q = \exp(-\pi K'/K)$, then one of the central theorems in the theory of elliptic functions asserts that [12, p. 101, Entry 6]

$$\varphi^2(q) = \frac{2}{\pi} K(k) = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right), \quad (3.2)$$

where $\varphi(q)$ in Ramanujan's notation (or $\vartheta_3(q)$ in the classical notation) denotes the classical theta function defined by

$$\varphi(q) = \sum_{j=-\infty}^{\infty} q^{j^2}. \quad (3.3)$$

Note that, in the notation (1.5), $q = q_2$. The second equality in (3.2) follows from expanding the integrand in a binomial series and integrating termwise. Let K, K', L , and L' denote complete elliptic integrals of the first kind associated with the moduli k, k', ℓ , and ℓ' , respectively. Suppose that, for some positive integer n ,

$$n \frac{K'}{K} = \frac{L'}{L}. \quad (3.4)$$

A modular equation of degree n is an equation involving k and ℓ that is induced by (3.4). Modular equations are always algebraic equations. An example of a modular equation of degree 7 may be found later in (9.18). Alternatively, by (3.2), (3.4) can be expressed in terms of hypergeometric functions. We often say that ℓ has degree n over k . Derivations of modular equations ultimately rest on (3.2). If we set $K/K' = \sqrt{n}$, so that $q = e^{-\pi\sqrt{n}}$, then $k(q)$ is called the *singular modulus*. The multiplier $m = m(q)$ is defined by

$$m := m(q) := \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k^2(q)\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \ell^2(q)\right)}. \quad (3.5)$$

We note here that $k^2(q)$ and $m(q)$ can be represented by [12, Entry 3, p. 98; Entry 24(ii), p. 214]

$$k^2(q) = 16q \frac{\psi^4(q^2)}{\varphi^4(q)} \quad \text{and} \quad m(q) = \frac{\varphi^2(q)}{\varphi^2(q^n)},$$

respectively, where $\varphi(q)$ is defined by (3.3) and

$$\psi(q) = \sum_{j=0}^{\infty} q^{j(j+1)/2}.$$

Thus, modular equations can also be written as theta function identities.

Ramanujan begins Section 13 of [55] with a special case of Clausen's formula [22, p. 178, Proposition 5.6(b)],

$$\frac{4K^2}{\pi^2} = \sum_{j=0}^{\infty} \frac{\left(\frac{1}{2}\right)_j^3}{(j!)^3} (2kk')^{2j} = {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, 1; (2kk')^2\right), \quad (3.6)$$

which can be found as Entry 13 of Chapter 11 in his second notebook [57] [11, p. 58]. Except for economizing notation, we now quote Ramanujan. "Hence we have

$$q^{1/3}(q^2; q^2)_{\infty}^4 = \left(\frac{1}{4}kk'\right)^{2/3} \sum_{j=0}^{\infty} \frac{\left(\frac{1}{2}\right)_j^3}{(j!)^3} (2kk')^{2j}, \quad (3.7)$$

where

$$(a; q)_{\infty} := (1-a)(1-aq)(1-aq^2) \cdots. \quad (3.8)$$

Differentiating both sides in (3.7) logarithmically with respect to k , we can easily shew that

$$1 - 24 \sum_{j=1}^{\infty} \frac{jq^{2j}}{1-q^{2j}} = (1-2k^2) \sum_{j=0}^{\infty} (3j+1) \frac{\left(\frac{1}{2}\right)_j^3}{(j!)^3} (2kk')^{2j}. \quad (3.9)$$

But it follows from

$$1 - \frac{3}{\pi\sqrt{n}} - 24 \sum_{j=1}^{\infty} \frac{j}{e^{2\pi j\sqrt{n}} - 1} = \left(\frac{K}{\pi}\right)^2 A(k) \quad (3.10)$$

where $A(k)$ is a certain type of algebraic number, that, when $q = e^{-\pi\sqrt{n}}$, n being a rational number, the left-hand side of (3.9) can be expressed in the form

$$A \left(\frac{2K}{\pi}\right)^2 + \frac{B}{\pi},$$

where A and B are algebraic numbers expressible by surds. Combining (3.6) and (3.9) in such a way as to eliminate the term $(2K/\pi)^2$, we are left with a series for $1/\pi$." He then gives the three examples (1.2)–(1.4).

Ramanujan's ideas will be described in more detail in Section 9. However, in closing this section, we note that the series on the left-hand sides of (3.9) and (3.10) is Ramanujan's Eisenstein series $P(q^2)$, with $q = e^{-\pi\sqrt{n}}$ in the latter instance, where

$$P(q) := 1 - 24 \sum_{j=1}^{\infty} \frac{jq^j}{1-q^j}, \quad |q| < 1. \quad (3.11)$$

Ramanujan's derivation of (3.10) arises firstly from the transformation formula for $P(q)$, which in turn is an easy consequence of the transformation formula for the Dedekind eta-function, given in (9.11) below. The second ingredient in deriving (3.10) is an identity for $nP(q^{2n}) - P(q^2)$ in terms of the moduli k and ℓ , where ℓ has degree n over k . Formula (3.7) follows from a standard theorem in elliptic functions that Ramanujan also recorded in his notebooks [57], [12, p. 124, Entry 12(iii)].

4. SARVADAMAN CHOWLA

Chowla's ideas reside in the classical theory of elliptic functions and are not unlike those that the Borweins employed several years later. We now briefly describe Chowla's [35], [36], [37, pp. 87–91, 116–119] approach. Using classical formulas of Cayley and Legendre relating the complete elliptic integrals K and E , he specializes them by setting $K/K' = \sqrt{n}$. He then defines

$$S_r := \sum_{j=0}^{\infty} j^r \frac{\left(\frac{1}{2}\right)_j^3}{(j!)^3} (2kk')^{2j} \quad (4.1)$$

and

$$T_r := \sum_{j=0}^{\infty} j^r \frac{\left(\frac{1}{4}\right)_j^2}{(j!)^2} (2kk')^{2j}. \quad (4.2)$$

Chowla then writes “Then it is known that, when $k \leq 1/\sqrt{2}$,”

$$\frac{2K}{\pi} = 1 + T_0 \quad \text{and} \quad \frac{4K^2}{\pi^2} = 1 + S_0. \quad (4.3)$$

Chowla does not give his source for either formula, but the second formula in (4.3), as noted above, is a special case of Clausen's formula (3.6). The first formula is a special case of Kummer's quadratic transformation [22, pp. 179–180], which was also known to Ramanujan. Each of the formulas of (4.3) is differentiated twice with respect to k , and, without giving details, Chowla concludes that if $K/K' = \sqrt{n}$, then

$$\begin{aligned} \frac{1}{K} &= a_1 + b_1 T_0 + c_1 T_1, \\ \frac{K}{\pi} &= d_1 T_1 + e_1 T_2, \\ \frac{1}{\pi} &= a_2 S_0 + b_2 S_1, \\ \frac{1}{K^2} &= a_3 + b_3 S_0 + c_3 S_1 + d_3 S_2, \end{aligned}$$

“where a_1, b_1, \dots are algebraic numbers.” He then sets $n = 3$ and $k = \sin(\pi/12)$ in each of the four formulas above to deduce, in particular, identity (1.2) from the second formula above.

5. R. WILLIAM GOSPER, JR.

As we indicated in the Introduction, in November, 1985, Gosper employed a lisp machine at Symbolics and Ramanujan's series (2.1) to calculate 17,526,100 digits of π , which at that time was a world record. Of the 17 series found by Ramanujan, this one converges the fastest, giving about 8 digits of π per term. At the time of Gosper's calculation, the world record for digits of π was about 16 million digits calculated by Y. Kanada. Before the Borwein brothers had later found a "conventional" proof of Ramanujan's series (2.1), they had shown that either (2.1) yields an exact formula for π or that it differs from π by more than $10^{-3000000}$. Thus, by demonstrating that his calculation of π agreed with that of Kanada, Gosper effectively had completed the Borweins' first proof of (2.1). However, Gosper's primary goal was not to eclipse Kanada's record but to study the (simple) continued fraction expansion of π for which he calculated 17,001,303 terms. In email letters [45] from February, 1992 and May, 1993, Gosper offered the following remarks on his calculations:

Of course, what the scribblers always censor is that the digits were a by-product. I wanted to change the object of the game away from meaningless decimal digits.

I used what I call a resumable matrix tower to exactly compute an enormous rational equal to the sum of a couple of million terms of Ramanujan's 99^{-4n} series. I then divided to form the binary integer = $\text{floor}(\pi 2^{58,000,000})$. I converted this to decimal and sent a summary to Kanada for comparison, and converted the binary fraction to a cf using an fft based scheme.

Gosper's world record was short lived, as in January, 1986, D. H. Bailey used an algorithm of the Borweins arising from a fourth order modular equation to compute 29,360,000 digits of π . Gosper's calculation of the continued fraction expansion of π was motivated by the fact that many important mathematical constants do not have interesting decimal expansions but do have interesting continued fraction expansions. That continued fraction expansions are considerably more interesting than decimal expansions is a view shared by the Chudnovsky brothers [41]. Continued fraction expansions can often be used to distinguish a constant from others, while decimal expansions likely will be unable to do so. For example, the simple continued fraction of e , namely,

$$e = 2 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{1 + \frac{1}{4 + \frac{1}{1 + \frac{1}{1 + \frac{1}{6 + \frac{1}{1 + \dots}}}}}}}}}}$$

has a pattern. On the contrary, taking a large random string of digits of e would not help one identify e . It is an open problem if the simple continued fraction of π , namely,

$$\pi = 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1 + \frac{1}{292 + \frac{1}{1 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{3 + \frac{1}{1 + \frac{1}{14 + \frac{1}{2 + \frac{1}{1 + \frac{1}{1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{1 + \frac{1}{84 + \frac{1}{2 + \frac{1}{1 + \frac{1}{1 + \frac{1}{15 + \frac{1}{3 + \frac{1}{13 + \frac{1}{1 + \frac{1}{4 + \frac{1}{2 + \frac{1}{6 + \dots}}}}}}}}}}}}}}}}}}}}}}}}}}}}}}$$

has a pattern.

Gosper also derived a hypergeometric-like series representation for π , namely,

$$\pi = \sum_{j=0}^{\infty} \frac{50j - 6}{\binom{3j}{j} 2^j}, \quad (5.1)$$

which can be used to calculate any particular binary digit of π . See a paper by G. Almkvist, C. Krattenthaler, and J. Petersson [1] for a proof of (5.1) as well as generalizations, which include the following theorem.

Theorem 5.1. *For each integer $k \geq 1$, there exists a formula*

$$\pi = \sum_{j=0}^{\infty} \frac{S_k(j)}{\binom{8kj}{4kj} (-4)^{kj}},$$

where $S_k(j)$ is a polynomial in j of degree $4k$ with rational coefficients.

6. JONATHAN AND PETER BORWEIN

One key to the work of both Ramanujan and Chowla in their derivations of formulas for $1/\pi$ is Clausen's formula for the square of a complete elliptic integral of the first kind or, by (3.2), for the square of the hypergeometric function ${}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; k^2)$. The aforementioned rendition (3.6) of Clausen's formula is not the most general version of Clausen's formula, namely,

$${}_2F_1^2\left(a, b; a + b + \frac{1}{2}; z\right) = {}_3F_2\left(2a, 2b, a + b; a + b + \frac{1}{2}, 2a + 2b; z\right). \quad (6.1)$$

Indeed, the work of many authors who have proved Ramanujan-like series for $1/\pi$ ultimately rests on special cases of (6.1). In particular, squares of certain other hypergeometric functions lead one to Ramanujan's alternative theories of elliptic functions.

A second key step is to find another formula for $(K/\pi)^2$, which also contains another term involving $1/\pi$. Combining the two formulas to eliminate the term $(K/\pi)^2$ then produces a hypergeometric-type series representation for $1/\pi$. Evidently, unaware of Chowla's earlier work, the Borweins proceeded in a similar fashion and used Legendre's relation [22, p. 24]

$$E(k)K'(k) + E'(k)K(k) - K(k)K'(k) = \frac{\pi}{2}$$

and other relations between elliptic integrals to produce such formulas.

Having derived a series representation for $1/\pi$, one now faces the problem of evaluating the moduli and elliptic integrals that appear in the formulas. When $q = e^{-\pi\sqrt{n}}$, then for certain positive integers n one can evaluate the requisite quantities. This leads us to the definition of the Ramanujan–Weber class invariants. After Ramanujan, set

$$\chi(q) := (-q; q^2)_{\infty}, \quad |q| < 1. \quad (6.2)$$

If n is a positive rational number and $q = e^{-\pi\sqrt{n}}$, then the class invariants G_n and g_n are defined by

$$G_n := 2^{-1/4} q^{-1/24} \chi(q) \quad \text{and} \quad g_n := 2^{-1/4} q^{-1/24} \chi(-q). \quad (6.3)$$

In the notation of H. Weber [62], $G_n = 2^{-1/4} \mathfrak{f}(\sqrt{-n})$ and $g_n = 2^{-1/4} \mathfrak{f}_1(\sqrt{-n})$. As mentioned in Section 3, $k_n := k(e^{-\pi\sqrt{n}})$ is called the *singular modulus*. In his voluminous

work on modular equations, Ramanujan sets $\alpha := k^2$ and $\beta := \ell^2$. Accordingly, we set $\alpha_n := k_n^2$. Because [12, p. 124, Entries 12(v), (vi)]

$$\chi(q) = 2^{1/6} \{\alpha(1-\alpha)/q\}^{-1/24} \quad \text{and} \quad \chi(-q) = 2^{1/6} \{\alpha(1-\alpha)^{-2}/q\}^{-1/24},$$

it follows from (6.3) that

$$G_n = \{4\alpha_n(1-\alpha_n)\}^{-1/24} \quad \text{and} \quad g_n = \{4\alpha_n(1-\alpha_n)^{-2}\}^{-1/24}. \quad (6.4)$$

Noting the arguments on the right-hand side of (3.6) when $q = e^{-\pi\sqrt{n}}$, we see the importance of (6.4).

The singular moduli and class invariants are actually algebraic numbers. In general, as n increases, the corresponding series for $1/\pi$ converges more rapidly. The series (2.2) is associated with the imaginary quadratic field $\mathbb{Q}(\sqrt{-163})$.

The Borweins' proofs of all 17 of Ramanujan's series for $1/\pi$ can be found in their book [22]. Their derivations arise from several general hypergeometric-like series representations for $1/\pi$ given in terms of singular moduli, class invariants, and complete elliptic integrals [22, pp. 181–184]. Another account of their work, but with fewer details, can be found in their paper [23] commemorating the centenary of Ramanujan's birth. Further celebrating the 100th anniversary of Ramanujan's birth, the Borweins derived further series for $1/\pi$ in [24]. The series in this paper correspond to imaginary quadratic fields with class number 2, with one of their series corresponding to $d = -427$ and yielding about 25 digits of π per term. In [28], the authors derived series for $1/\pi$ arising from fields with class number 3, with a series corresponding to $d = -907$ yielding about 37 or 38 digits of π per term. Their record is a series associated with a field of class number 4 giving about 50 digits of π per term; here, $d = -1555$ [27]. The latter paper gives the details of what we have written in this paragraph.

The Borweins have done an excellent job of communicating their work to a wide audience. Besides their paper [27], see their paper in the *American Mathematical Monthly* [29], with D. H. Bailey, on computing π , especially via work of Ramanujan, and their delightful paper in the *Scientific American* [25], which has been reprinted in [21, pp. 187–199].

7. DAVID AND GREGORY CHUDNOVSKY

In our Introduction we mentioned that the Chudnovsky brothers, Gregory and David, used (2.2) to calculate over 2 billion digits of π . They had first used (2.2) to calculate 1,130,160,664 digits of π in the fall of 1989 on a “borrowed” computer. They then built their own computer, “m zero,” described colorfully in [54], and set a world record of 2,260,321,336 digits of π . The world record for digits of π has been broken several times since then, and since it is not the purpose of this paper to delineate this computational history, we refrain from mentioning further records.

The Chudnovskys, among others, have extensively examined their calculations for patterns. It is a long outstanding conjecture that π is *normal*. In particular, if N denotes the number of calculated digits of π , then for each k , $0 \leq k \leq 9$,

$$\lim_{N \rightarrow \infty} \frac{\# \text{ of appearances of } k \text{ in the first } N \text{ digits}}{N} = \frac{1}{10}.$$

The Chudnovskys' calculations, and all subsequent calculations of Kanada, lend credence to this conjecture. As a consequence, the average of the digits over a long interval should be approximately 4.5. The Chudnovskys found that for the first billion digits the average stays a bit on the high side, while for the next billion digits, the average hovers a bit on the low side. Their paper [41] gives an interesting statistical analysis of the digits up to one billion. For example, strings of consecutive digits of maximal lengths between 8 and 10 occur for each digit.

The Chudnovskys deduced (2.2) from a general series representation for $1/\pi$ for which we need to make several definitions in order to describe. For $\tau \in \mathcal{H} = \{\tau : \text{Im } \tau > 0\}$ and each positive integer k , the Eisenstein series $E_{2k}(\tau)$ is defined by

$$E_{2k}(\tau) := 1 - \frac{4k}{B_{2k}} \sum_{j=1}^{\infty} \sigma_{2k-1}(j)q^j, \quad q = e^{\pi i \tau}, \quad (7.1)$$

where B_k , $k \geq 0$, denotes the k th Bernoulli number and $\sigma_k(n) = \sum_{d|n} d^k$. Klein's absolute modular J -invariant is defined by

$$J(\tau) := \frac{E_4^3(\tau)}{E_4^3(\tau) - E_6^2(\tau)}, \quad \tau \in \mathcal{H}. \quad (7.2)$$

It is well known that if $\alpha(q) := k^2(q)$, where k is the modulus, then [12, pp. 126–127, Entry 13]

$$J(2\tau) = \frac{4(1 - \alpha(q) + \alpha^2(q))^3}{27\alpha^2(q)(1 - \alpha(q))^2}. \quad (7.3)$$

Thus, (6.4) and (7.3) show that, when $q = e^{-\pi\sqrt{n}}$, singular moduli, class invariants, and the modular J -invariant are intimately related. Now define

$$s_2(\tau) := \frac{E_4(\tau)}{E_6(\tau)} \left(E_2(\tau) - \frac{3}{\pi \text{Im } \tau} \right).$$

We are now ready to state the Chudnovskys' main formula [42, p. 122]. If $\tau = (1 + \sqrt{-d})/2$, then

$$\sum_{\mu=0}^{\infty} \left\{ \frac{1}{6}(1 - s_2(\tau)) + \mu \right\} \frac{(6\mu)!}{(3\mu)! \mu!^3} \frac{1}{1728^\mu J^\mu(\tau)} = \frac{\sqrt{-J(\tau)}}{\pi} \frac{1}{\sqrt{d(1 - J(\tau))}}. \quad (7.4)$$

The Chudnovskys' series (2.2) is the special case $d = 163$ of (7.4).

The Chudnovsky brothers developed and extended Ramanujan's ideas in directions different from those of other authors. They obtained hypergeometric-like representations for other transcendental constants and proved, for example, that

$$\frac{\Gamma\left(\frac{1}{3}\right)}{\pi} \quad \text{and} \quad \frac{\Gamma^2\left(\frac{1}{24}\right)}{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{1}{4}\right)}$$

are transcendental [40]. They extended their analysis to linear forms of logarithms. Their advances involve the “second” solution of the hypergeometric differential equation. Recall from the theory of linear differential equations that ${}_2F_1(a, b; c; x)$ is a solution of a certain second order linear differential equation with a regular singular

point at the origin [5, p. 1]. A second linearly independent solution is generally not analytic at the origin, and in [41] and [42, pp. 124–126], the Chudnovsky brothers establish new hypergeometric series identities involving the latter function. Their identities in this paper lead to hypergeometric-like series representations for π , including Gosper's formula (5.1). In [43], the authors provide a lengthy list of such examples, including

$$45\pi + 644 = \sum_{j=0}^{\infty} \frac{8^j(430j^2 - 6240j - 520)}{\binom{4j}{j}}.$$

The Chudnovsky brothers have also employed series for $1/\pi$ to derive theorems on transcendence measures $\mu(\alpha)$, which are defined by

$$\mu(\alpha) := \inf \left\{ \mu > 0 : 0 < \left| \alpha - \frac{p}{q} \right| < \frac{1}{q^\mu} \text{ has finitely many solutions } \frac{p}{q} \in \mathbb{Q} \right\}.$$

By the famous Thue–Siegel–Roth Theorem [6, p. 66],

$$\mu(\alpha) \begin{cases} = 1, & \text{if } \alpha \text{ is rational,} \\ = 2, & \text{if } \alpha \text{ is algebraic but not rational,} \\ \geq 2, & \text{if } \alpha \text{ is transcendental.} \end{cases}$$

Although the Chudnovskys can obtain irrationality measures for various constants, the one they obtain for π is not as good as one would like. Currently, the world record for the irrationality measure of π is held by M. Hata [52], who proved that $\mu(\pi) \leq 8.016045\dots$. Their methods are much better in obtaining irrationality measures for expressions such as $\pi/\sqrt{640320}$, arising in their series (2.2). See also a paper by W. Zudilin [64].

8. RAMANUJAN'S CUBIC CLASS INVARIANT AND HIS ALTERNATIVE THEORIES

In Sections 3, 4, 6, and 7, we emphasized how Ramanujan–Weber class invariants and singular moduli were of central importance for Ramanujan and others who followed in deriving series for $1/\pi$. We also stressed in Section 1, in particular, in the discourse after (1.5), that Ramanujan's remarkable observation of replacing the classical hypergeometric function ${}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x)$ by ${}_2F_1(\frac{1}{r}, \frac{r-1}{r}; 1; x)$, $r = 3, 4, 6$, leads to new and beautiful alternative theories. On the top of the page 212 in his lost notebook [58], Ramanujan defines a cubic class invariant λ_n (i.e., $r = 3$ in (1.5)), which is an analogue of the Ramanujan–Weber classical invariants G_n and g_n defined in (6.3). Define Ramanujan's function

$$f(-q) := (q; q)_\infty, \quad |q| < 1, \quad (8.1)$$

where $(a; q)_\infty$ is defined in (3.8), and the Dedekind eta-function $\eta(\tau)$

$$\eta(\tau) := e^{2\pi i\tau/24} \prod_{j=1}^{\infty} (1 - e^{2\pi i j\tau}) =: q^{1/24} f(-q), \quad (8.2)$$

where $q = e^{2\pi i\tau}$ and $\text{Im } \tau > 0$. Then Ramanujan's cubic class invariant λ_n is defined by

$$\lambda_n = \frac{1}{3\sqrt{3}} \frac{f^6(q)}{\sqrt{q}f^6(q^3)} = \frac{1}{3\sqrt{3}} \left(\frac{\eta\left(\frac{1+i\sqrt{n/3}}{2}\right)}{\eta\left(\frac{1+i\sqrt{3n}}{2}\right)} \right)^6, \quad (8.3)$$

where $q = e^{-\pi\sqrt{n/3}}$, i.e., $\tau = \frac{1}{2}i\sqrt{n/3}$.

Chan, Liaw, and Tan [32] established a general series representation for $1/\pi$ in terms of λ_n that is analogous to the general formulas of the Borweins and Chudnovskys in terms of the classical class invariants. To state this general formula, we first need some definitions. Define

$$\frac{1}{\alpha^*(q)} := -\frac{1}{27q} \frac{f^{12}(q)}{f^{12}(q^3)} + 1. \quad (8.4)$$

Thus, when $q = e^{-\pi\sqrt{n/3}}$ and $\alpha_n^* := \alpha^*(e^{-\pi\sqrt{n/3}})$, (8.3) and (8.4) imply that

$$\frac{1}{\alpha_n^*} = 1 - \lambda_n^2.$$

In analogy with (3.5), define the multiplier $\mathbf{m}(q)$ by

$$\mathbf{m}(q) := \mathbf{m}(\alpha^*, \beta^*) := \frac{{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; \alpha^*\right)}{{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; \beta^*\right)},$$

where $\beta^* = \alpha^*(q^n)$. We are now ready to state the general representation of $1/\pi$ derived by Chan, Liaw, and Tan [32, p. 102, Theorem 4.2].

Theorem 8.1. *For $n \geq 1$, let*

$$a_n = -\frac{\alpha_n^*(1 - \alpha_n^*)}{\sqrt{n}} \frac{d\mathbf{m}(\alpha^*, \beta^*)}{d\alpha^*} \Big|_{\alpha^*=1-\alpha_n^*, \beta^*=\alpha_n^*},$$

$$b_n = 1 - 2\alpha_n^*,$$

and

$$H_n = 4\alpha_n^*(1 - \alpha_n^*).$$

Then

$$\frac{1}{\pi} \sqrt{\frac{3}{n}} = \sum_{j=0}^{\infty} (a_n + b_n j) \frac{\left(\frac{1}{2}\right)_j \left(\frac{1}{3}\right)_j \left(\frac{2}{3}\right)_j}{(j!)^3} H_n^j. \quad (8.5)$$

We give one example. Let $n = 9$; then $\alpha_9^* = \frac{9}{8}$. Then, without providing further details,

$$\frac{4}{\pi\sqrt{3}} = \sum_{j=0}^{\infty} (5j + 1) \frac{\left(\frac{1}{2}\right)_j \left(\frac{1}{3}\right)_j \left(\frac{2}{3}\right)_j}{(j!)^3} \left(-\frac{9}{16}\right)^j,$$

which was discovered by Chan, Liaw, and Tan [32, p. 95].

Another general series representation for $1/\pi$ in the alternative theories of Ramanujan was devised by Berndt and Chan [18, p. 88, Eq. (5.80)]. We will not state this

formula and all the requisite definitions, but let it suffice to say that the formula involves Ramanujan's Eisenstein series $P(q)$, $Q(q)$, and $R(q)$ at the argument $q = -e^{-\pi\sqrt{n}}$ and the modular j -invariant, defined by $j(\tau) = 1728J(\tau)$, where $J(\tau)$ is defined by (7.2). In particular,

$$j\left(\frac{3 + \sqrt{-3n}}{2}\right) = -27 \frac{(\lambda_n^2 - 1)(9\lambda_n^2 - 1)^3}{\lambda_n^2},$$

a proof of which can be found in [17]. The hypergeometric terms are of the form

$$\frac{\left(\frac{1}{2}\right)_j \left(\frac{1}{6}\right)_j \left(\frac{5}{6}\right)_j}{(j!)^3}.$$

Berndt and Chan used their general formula to calculate a series for $1/\pi$ that yields about 73 or 74 digits of π per term.

Lastly, we conclude this section by remarking that Berndt, Chan, and Liaw [19] have derived series representations for $1/\pi$ that fall under the umbrella of Ramanujan's quartic theory of elliptic functions. Because the quartic theory is intimately connected with the classical theory, their general formulas [19, p. 144, Theorem 4.1] involve the classical invariants G_n and g_n in their summands. Not surprisingly, the hypergeometric terms are of the form

$$B_j := \frac{\left(\frac{1}{2}\right)_j \left(\frac{1}{4}\right)_j \left(\frac{3}{4}\right)_j}{(j!)^3}.$$

The simplest example arising from their theory is given by [19, p. 145]

$$\frac{9}{2\pi} = \sum_{j=0}^{\infty} B_j(7j+1) \left(\frac{32}{81}\right)^j.$$

9. THE PRESENT AUTHORS AS DISCIPLES OF RAMANUJAN

As mentioned in Section 2, the first two authors were inspired by the third author to continue the development of Ramanujan's thoughts. In their first paper [7], Baruah and Berndt employed Ramanujan's ideas in the classical theory of elliptic functions to prove 13 of Ramanujan's original formulas and many new ones as well. In [8], they utilized Ramanujan's cubic and quartic theories to establish five of Ramanujan's 17 formulas in addition to some new representations. Lastly, in [9], motivated by the work of J. Guillera, described briefly in Section 10 below, the first two authors extended Ramanujan's ideas to derive hypergeometric-like series representations for $1/\pi^2$. For example,

$$\frac{24}{\pi^2} = \sum_{\mu=0}^{\infty} (44571654400\mu^2 + 5588768408\mu + 233588841) B_{\mu} \left(\frac{1}{994}\right)^{\mu+1},$$

where

$$B_{\mu} = \sum_{\nu=0}^{\mu} \frac{\left(\frac{1}{4}\right)_{\nu} \left(\frac{1}{4}\right)_{\mu-\nu} \left(\frac{1}{2}\right)_{\nu} \left(\frac{1}{2}\right)_{\mu-\nu} \left(\frac{3}{4}\right)_{\nu} \left(\frac{3}{4}\right)_{\mu-\nu}}{\nu!^3 (\mu-\nu)!^3}.$$

In Section 3, we defined Ramanujan's Eisenstein series $P(q)$ in (3.11) and offered several definitions from Ramanujan's theories of elliptic functions in giving a brief

introduction to Ramanujan's ideas. Here we highlight the role of $P(q)$ in more detail before giving a complete proof of (1.3). Because these three series representations (1.2)–(1.4) can also be found in Ramanujan's lost notebook, our proof here is similar to that given in [3, Chapter 15].

Following Ramanujan, set

$$z := {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right). \quad (9.1)$$

The two most important ingredients in our derivations are Ramanujan's representation for $P(q^2)$ given by [12, p. 120, Entry 9(iv)]

$$P(q^2) = (1 - 2x)z^2 + 6x(1 - x)z\frac{dz}{dx} \quad (9.2)$$

and Clausen's formula (3.6), which, using (9.1), we restate in the form

$$z^2 = {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, 1; X\right) = \sum_{j=0}^{\infty} A_j X^j, \quad (9.3)$$

where, as in (1.1),

$$A_j := \frac{\left(\frac{1}{2}\right)_j^3}{j!^3} \quad \text{and} \quad X := 4x(1 - x). \quad (9.4)$$

From (9.3) and (9.4),

$$2z\frac{dz}{dx} = \sum_{j=0}^{\infty} A_j j X^{j-1} \cdot 4(1 - 2x). \quad (9.5)$$

Hence, from (9.2), (9.3), (9.5), and (9.4),

$$\begin{aligned} P(q^2) &= (1 - 2x) \sum_{j=0}^{\infty} A_j X^j + 3(1 - 2x) \sum_{j=0}^{\infty} A_j j X^j \\ &= \sum_{j=0}^{\infty} \{(1 - 2x) + 3(1 - 2x)j\} A_j X^j. \end{aligned} \quad (9.6)$$

For $q := e^{-\pi\sqrt{n}}$, recall (9.1) and set

$$x_n = k^2(e^{-\pi\sqrt{n}}), \quad z_n := {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x_n\right) \quad (9.7)$$

and

$$X_n = 4x_n(1 - x_n). \quad (9.8)$$

For later use, we note that [3, p. 375]

$$1 - x_n = x_{1/n} \quad \text{and} \quad z_{1/n} = \sqrt{n}z_n. \quad (9.9)$$

With the use of (9.7) and (9.8), (9.6) takes the form

$$\begin{aligned} P(e^{-2\pi\sqrt{n}}) &= \sum_{j=0}^{\infty} \{(1 - 2x_n) + 3(1 - 2x_n)j\} A_j X_n^j \\ &= (1 - 2x_n)z_n^2 + 3 \sum_{j=0}^{\infty} (1 - 2x_n)j A_j X_n^j. \end{aligned} \quad (9.10)$$

In order to utilize (9.10), we require two different formulas, each involving both $P(q^2)$ and $P(q^{2n})$, where n is a positive integer. The first comes from a transformation formula for $P(q)$, which in turn arises from the transformation formula for $f(-q)$ defined in (8.1) or the Dedekind eta function defined in (8.2), and is for general n . This transformation formula is given by [12, p. 43, Entry 27(iii)]

$$e^{-\alpha/12}\alpha^{1/4}f(-e^{-2\alpha}) = e^{-\beta/12}\beta^{1/4}f(-e^{-2\beta}), \quad (9.11)$$

where $\alpha\beta = \pi^2$, with α and β both positive. Taking the logarithm of both sides of (9.11), we find that

$$-\frac{\alpha}{12} + \frac{1}{4}\log\alpha + \sum_{j=1}^{\infty}\log(1 - e^{-2j\alpha}) = -\frac{\beta}{12} + \frac{1}{4}\log\beta + \sum_{j=1}^{\infty}\log(1 - e^{-2j\beta}). \quad (9.12)$$

Differentiating both sides of (9.12) with respect to α , we deduce that

$$-\frac{1}{12} + \frac{1}{4\alpha} + \sum_{j=1}^{\infty}\frac{2je^{-2j\alpha}}{1 - e^{-2j\alpha}} = \frac{\beta}{12\alpha} - \frac{1}{4\alpha} - \sum_{j=1}^{\infty}\frac{(2j\beta/\alpha)e^{-2j\beta}}{1 - e^{-2j\beta}}. \quad (9.13)$$

Multiplying both sides of (9.13) by 12α and rearranging, we arrive at

$$6 - \alpha \left(1 - 24 \sum_{j=1}^{\infty} \frac{je^{-2j\alpha}}{1 - e^{-2j\alpha}} \right) = \beta \left(1 - 24 \sum_{j=1}^{\infty} \frac{je^{-2j\beta}}{1 - e^{-2j\beta}} \right). \quad (9.14)$$

Setting $\alpha = \pi/\sqrt{n}$, so that $\beta = \pi\sqrt{n}$, recalling the definition (3.11) of $P(q)$, and rearranging slightly, we see that (9.14) takes the shape

$$\frac{6\sqrt{n}}{\pi} = P(e^{-2\pi/\sqrt{n}}) + nP(e^{-2\pi\sqrt{n}}). \quad (9.15)$$

This is the first desired formula.

The second gives representations for Ramanujan's function [55], [56, pp. 33–34]

$$f_n(q) := nP(q^{2n}) - P(q^2) \quad (9.16)$$

for *certain* positive integers n . (Ramanujan [55], [56, pp. 33–34] used the notation $f(n)$ instead of $f_n(q)$.) In [55], Ramanujan recorded representations for $f_n(q)$ for 12 values of n , but he gave no indication of how these might be proved. These formulas are also recorded in Chapter 21 of Ramanujan's second notebook [57], and proofs may be found in [12].

We now give the details for our proof of (1.3), which was clearly a favorite of Gabriella Montez, the precocious student in "High School Musical." Unfortunately, we do not know whether she possessed a proof of her own. We restate (1.3) here for convenience.

Theorem 9.1. *If A_j , $j \geq 0$, is defined by (9.4), then*

$$\frac{16}{\pi} = \sum_{j=0}^{\infty} (42j + 5)A_j \frac{1}{26^j}. \quad (9.17)$$

Proof. The identity (9.17) is connected with modular equations of degree 7. Thus, our first task is to calculate the singular modulus x_7 . To that end, we begin with a modular equation of degree 7

$$\{x(q)x(q^7)\}^{1/8} + \{(1-x(q))(1-x(q^7))\}^{1/8} = 1, \quad (9.18)$$

due to C. Guetzlaff in 1834 but rediscovered by Ramanujan in Entry 19(i) of Chapter 19 of his second notebook [57], [12, p. 314]. In the notation of our definition of a modular equation after (3.4), we have set $x(q) = k^2(q)$, and so $x(q^7) = \ell^2(q)$. Set $q = e^{-\pi/\sqrt{7}}$ in (9.18) and use (9.9) and (9.8) to deduce that

$$2\{x_7(1-x_7)\}^{1/8} = 1 \quad \text{and} \quad X_7 = \frac{1}{2^6}. \quad (9.19)$$

Ramanujan calculated the singular modulus x_7 in his first notebook [57], [14, p. 290], from which, or from (9.19), we easily can deduce that

$$1 - 2x_7 = \frac{3\sqrt{7}}{8}. \quad (9.20)$$

In the notation (9.16), from either [55], [56, p. 33] or [12, p. 468, Entry 5(iii)],

$$f_7(q) = 3z(q)z(q^7) \left(1 + \sqrt{x(q)x(q^7)} + \sqrt{(1-x(q))(1-x(q^7))}\right). \quad (9.21)$$

Putting $q = e^{-\pi/\sqrt{7}}$ in (9.21) and employing (9.9) and (9.19), we find that

$$f_7(e^{-\pi/\sqrt{7}}) = 3\sqrt{7} \left(1 + 2\sqrt{x_7(1-x_7)}\right) z_7^2 = 3\sqrt{7} \cdot \frac{9}{8} z_7^2. \quad (9.22)$$

Letting $n = 7$ in (9.15) and (9.10), and using (9.20), we see that

$$\frac{6\sqrt{7}}{\pi} = P(e^{-2\pi/\sqrt{7}}) + 7P(e^{-2\pi\sqrt{7}}) \quad (9.23)$$

and

$$\begin{aligned} P(e^{-2\pi\sqrt{7}}) &= (1 - 2x_7)z_7^2 + 3 \sum_{j=0}^{\infty} (1 - 2x_7)jA_j X_7^j \\ &= \frac{3\sqrt{7}}{8} z_7^2 + \frac{9\sqrt{7}}{8} \sum_{j=0}^{\infty} \frac{jA_j}{2^{6j}}, \end{aligned} \quad (9.24)$$

respectively. Eliminating $P(e^{-2\pi/\sqrt{7}})$ from (9.22) and (9.23) and putting the resulting formula for $P(e^{-2\pi\sqrt{7}})$ in (9.24), we find that

$$\frac{3\sqrt{7}}{7\pi} + \frac{27\sqrt{7}}{16 \cdot 7} z_7^2 = \frac{3\sqrt{7}}{8} z_7^2 + \frac{9\sqrt{7}}{8} \sum_{j=0}^{\infty} \frac{jA_j}{2^{6j}},$$

which upon simplification with the use of (9.3) yields (9.17). \square

10. JESÚS GUILLERA

A discrete function $A(n, k)$ is hypergeometric if

$$\frac{A(n+1, k)}{A(n, k)} \quad \text{and} \quad \frac{A(n, k+1)}{A(n, k)}$$

are both rational functions. A pair of functions $F(n, k)$ and $G(n, k)$ is said to be a WZ pair (after H. S. Wilf and D. Zeilberger) if F and G are hypergeometric and

$$F(n+1, k) - F(n, k) = G(n, k+1) - G(n, k).$$

In this case, H. Wilf and D. Zeilberger [53] showed that there exists a rational function $C(n, k)$ such that

$$G(n, k) = C(n, k)F(n, k).$$

The function $C(n, k)$ is called a certificate of (F, G) . Defining

$$H(n, k) = F(n+k, n+1) + G(n, n+k),$$

Wilf and Zeilberger showed that

$$\sum_{n=0}^{\infty} H(n, 0) = \sum_{n=0}^{\infty} G(n, 0).$$

Ekhad (Zeilberger's computer) and Zeilberger [44] were the first to use this method to derive a one page proof of the representation

$$\frac{2}{\pi} = \sum_{j=0}^{\infty} (-1)^j (4j+1) \frac{\left(\frac{1}{2}\right)_j^3}{(j!)^3}. \quad (10.1)$$

The identity (10.1) was first proved by G. Bauer in 1859 [10]. Ramanujan recorded (10.1) as Example 14 in Section 7 of Chapter 10 in his second notebook [57], [11, pp. 23–24]. Further references can be found in [7].

Motivated by this work, J. Guillera [46] found many new WZ-pairs (F, G) and derived new series not only for $1/\pi$ but for $1/\pi^2$ as well. One of his most elegant formulas is

$$\frac{128}{\pi^2} = \sum_{j=0}^{\infty} (-1)^j \binom{2j}{j}^5 \frac{(820j^2 + 180j + 13)}{2^{10j}}.$$

Subsequently, Guillera empirically discovered many series of the type

$$\frac{A}{\pi} = \sum_{j=0}^{\infty} c_j \frac{Bj^2 + Dj + E}{H^j}.$$

Most of the series he discovered cannot be proved by the WZ-method; it appears that the WZ-method is only applicable to those series for $1/\pi$ when H is a power of 2. An example of Guillera's series which remains to be proved is [47]

$$\frac{128\sqrt{5}}{\pi^2} = \sum_{j=0}^{\infty} (-1)^j \frac{\left(\frac{1}{2}\right)_j \left(\frac{1}{3}\right)_j \left(\frac{2}{3}\right)_j \left(\frac{1}{6}\right)_j \left(\frac{5}{6}\right)_j}{(j!)^5 80^{3j}} (5418j^2 + 693j + 29).$$

For further Ramanujan-like series for $1/\pi^2$, see Zudilin's papers [65], [66].

11. RECENT DEVELOPMENTS

We have emphasized in this paper that Clausen's formula (6.1) is an essential ingredient in most proofs of Ramanujan-type series representations for $1/\pi$. However, there are other kinds of series for $1/\pi$ that do not depend upon Clausen's formula. One such series discovered by Takeshi Sato [60] is given by

$$\frac{1}{\pi} \frac{\sqrt{15}}{120(4\sqrt{5}-9)} = \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\mu} \binom{\mu}{\nu}^2 \binom{\mu+\nu}{\nu}^2 \left(\frac{1}{2} - \frac{3}{20}\sqrt{5} + \mu\right) \left(\frac{\sqrt{5}-1}{2}\right)^{12\mu}. \quad (11.1)$$

In unpublished work, Sato [61] derived a more complicated series for $1/\pi$ that yields approximately 97 digits of π per term. A companion to (11.1), which was derived by a new method devised by the third author, S. H. Chan, and Z.-G. Liu [30], is given by

$$\frac{8}{\sqrt{3}\pi} = \sum_{\mu=0}^{\infty} a_{\mu} \frac{5\mu+1}{64^{\mu}}, \quad (11.2)$$

where

$$a_{\mu} := \sum_{\nu=0}^{\mu} \binom{2\mu-2\nu}{\mu-\nu} \binom{2\nu}{\nu} \binom{\mu}{\nu}^2. \quad (11.3)$$

We cite three further new series arising from this new method. The first is another companion of (11.2), which arises from recent work of Chan and H. Verrill [34] (after the work of Almkvist and Zudilin [2]), and is given by

$$\frac{9}{2\sqrt{3}\pi} = \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\lfloor \frac{\mu}{3} \rfloor} (-1)^{\mu-\nu} 3^{\mu-3\nu} \binom{\mu}{3\nu} \binom{\mu+\nu}{\nu} \frac{(3\nu)!}{(\nu!)^3} (4\mu+1) \left(\frac{1}{81}\right)^{\mu}.$$

The second is from a paper by Chan and K. P. Loo [33] and takes the form

$$\frac{2\sqrt{3}(3+2\sqrt{2})}{9\pi} = \sum_{\mu=0}^{\infty} C_{\mu} \left(\mu+1 - \frac{2}{3}\sqrt{2}\right) \left(-1 + \frac{3}{4}\sqrt{2}\right)^{\mu},$$

where

$$C_{\mu} = \sum_{\nu=0}^{\mu} \left\{ \sum_{j=0}^{\nu} \binom{\nu}{j}^3 \sum_{i=0}^{\mu-\nu} \binom{\mu-\nu}{i}^3 \right\}.$$

The third was derived by Y. Yang [63] and takes the shape

$$\frac{18}{\pi\sqrt{15}} = \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\mu} \binom{\mu}{\nu}^4 \frac{4\mu+1}{36^{\nu}}.$$

Arising out of his work with Mahler measures and new transformation formulas for ${}_5F_4$ series, M. D. Rogers [59, Corollary 3.2] has also discovered series for $1/\pi$ in the

spirit of the formulas above. For example, if a_μ is defined by (11.3), then

$$\frac{2}{\pi} = \sum_{\mu=0}^{\infty} (-1)^\mu a_\mu \frac{3\mu + 1}{32^\mu}.$$

This series was also independently discovered by Chan and Verrill [34].

According to W. Zudilin [67], G. Gourevich, using an integers relation algorithm, empirically discovered a hypergeometric-like series for $1/\pi^3$, namely,

$$\frac{32}{\pi^3} = \sum_{\mu=0}^{\infty} \frac{\left(\frac{1}{2}\right)_\mu^7}{(\mu!)^7 2^{6\mu}} (168\mu^3 + 76\mu^2 + 14\mu + 1).$$

This series and the search for further series representations for $1/\pi^m$, $m \geq 2$, are described in a paper by D. H. Bailey and J. M. Borwein [4].

We mentioned briefly in Section 5 that (5.1) could be used to determine any particular binary digit of π . It is likely that Ramanujan's series for $1/\pi$ can be used for the same purpose. To that end, let

$$P_{N,t} = \frac{1}{2^2 \cdot 3 \cdot 5^3 \sqrt{3}} \sum_{j=0}^{N+t} (14151j + 827) \frac{\left(\frac{1}{3}\right)_j \left(\frac{2}{3}\right)_j \left(\frac{1}{2}\right)_j}{(j!)^3} \left(-\frac{1}{250000}\right)^j.$$

Then it appears that the $(6N - 5)$ -th to $6N$ -th decimal places of $1/\pi$ can be obtained from the first six decimal places of $10^{6(N-1)} P_{N, \lfloor N/6 \rfloor}$. For example, when $N = 2$, we can obtain the 7-th to 12-th decimal places of $1/\pi$ from the first six decimal places of

$$\frac{1116441013229129}{6075000000} \sqrt{3}.$$

12. CONCLUSION

One test of “good” mathematics is that it should generate more “good” mathematics. Readers have undoubtedly concluded that Ramanujan's original series for $1/\pi$ have sown the seeds for an abundant crop of “good” mathematics.

13. ACKNOWLEDGMENTS

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